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MAGNIFYING ELEMENTS IN THE MONOID OF ALL PARTIAL TRANSFORMATIONS PRESERVING AN EQUIVALENCE RELATION AND A PARTITION

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Abstract. *An element a of a semigroup S is called a left* (*right*) *magnifying element if there exists a proper subset M of S satisfying $aM = S$ $(Ma = S)$. Let E be an equivalence relation and $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a nonempty set X. We consider the set $P_E(X, \mathcal{P}) = \{\alpha \in P(X) \mid \forall (x, y) \in E, (x\alpha, y\alpha) \in E$ and $X_i\alpha \subseteq X_i$ for all $i \in \Lambda\}$, which is a submonoid of $P(X)$, the set of all maps from A to X where $A \subseteq X$. The main propose of this paper is to establish the necessary and sufficient conditions for elements in $P_E(X, \mathcal{P})$ to be a left or right magnifying element. The exposition of our results provides the conceptual and practical application in establishing magnifying elements in some submonoid of $P(X)$. Furthermore, the characterization of these elements in generalized transformation semigroup is able to be a standard tool of studying more complex transformation semigroups.*

Keywords: Magnifying elements, Partial transformation semigroups, Equivalence relations, Partitions

1. **Introduction.** Let $P(X)$ be the set of all maps from A to a nonempty set X where $A \subseteq X$. It is well-known that $P(X)$ is a semigroup under the composition of functions with identity map id_X or a monoid of all partial transformations on a nonempty set X. Recall that an element a of a semigroup S is called a left (right) magnifying element if there exists a proper subset M of S satisfying $aM = S$ $(Ma = S)$. This notion was established by Ljapin [1] in 1963. The principal reason for establishing this element in a semigroup is applying the useful properties if the semigroup with unit contains magnifying elements. This semigroup has many properties which is noted in [1], for example, every magnifying element in the semigroup with unit is regular, an infinite monogenic semigroup can be generated by every magnifying element in the semigroup with unit. Moreover, magnifying elements property coincided with the property of invertibility of that elements, e.g., every right magnifying element of a semigroup is left invertible but not right invertible, every left magnifying element of a semigroup is right invertible but

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not left invertible. Actually, many researchers are interested in many elements in a semigroup such as ideals, regulars, inverse elements, and idempotents. The study of these elements had a long and fruitful idea in researches of semigroups. For instance, Chinram and Gaketem studied essential (m, n)-ideal and essential fuzzy (m, n)-ideals in a semigroup which play the important roles in studying of semigroups [12]. So it is also worth studying magnifying elements in a semigroup, especially, transformation semigroups since it appears everywhere in studying of mathematics. It is interesting and natural. As the usual composition of transformations is associative, the set of transformations, respect to the composition, forms a semigroup. Among all transformation semigroups, one that is much interesting is the partial transformation semigroup which is large and important enough in this field to be studied.

The investigations of magnifying elements in a semigroup were formed about 30 years ago after the element was mentioned. In 1992, Catino and Migliorini [2] showed that for a bisimple monoid S, either S is a group or S contains left and right magnifying elements. Furthermore, they pointed out that in a monoid the existence of left magnifying elements implies the existence of right magnifying elements, and vice versa. Two years later, the conditions for elements to be a magnifying element in any submonoid of the full transformation monoid were set up by Magill [3]. Recently, Luangchaisri et al. [5] characterized left and right magnifying elements in $P(X)$, which generalized those by Magill given in [3]. In [4], Prakitsri determined the existence of left and right magnifying elements in the linear transformation semigroups with infinite nullity and co-rank. Many authors have extensively studied the transformation monoids that preserve an equivalence relation in many aspects, e.g., regularity, Green's equivalences, and natural partial orders (see, for example, [6], [7], [8], [9] and [10]). In 2016, Purisang and Rakbud [11] investigated the regularity of some submonoids of $P(X)$ which is defined by a partition on X. In 2018, Chinram et al. [13] examined the magnifying elements in the monoid of all full transformations which is a submonoid of $P(X)$ preserving a partition on X.

We are motivated not only by its roles in the semigroup theory but also by our strong belief that these elements in generalized transformation semigroup are to be able to be standard tools for dealing with real world problems. Let us consider the following situation. The company plans to increase the efficiency of the organization. There are many tasks to do but the company has the only one leader. The leader wants to find the representatives to do all tasks perfectly without the laborious extravagance. Ideally, we assume that one product needs infinite methods, both the leader and each employee have to weigh their capabilities or satisfactions to each method by $1, 2, 3, \ldots$ under the conditions each method must have only one weight but some methods can have the same weight. From this idea, the set of employee forms a semigroup of full transformations on a set $\mathbb{N}$. If the capable function of the leader satisfying the conditions to be a left or right magnifying element, then we can choose some proper set of all employees to work having a performance as good as all employee. In case some of the employees cannot do some methods and hence we do not care if these employees do not weigh some methods. Then the set of employee form a semigroups of the partial transformations on a set $\mathbb{N}$.

Previously, the authors published the conditions for elements being left or right magnifying in the monoid of all full transformations preserving both an equivalence relation and a partition on the set X in [14], and in the monoid of partial transformations preserving an equivalence relation in [15]. However, no one has yet studied magnifying elements in $P(X)$ preserving both an equivalence relation and a partition on the set X. Consequently, we will study magnifying elements in the semigroup of all partial transformations preserving an equivalence relation E and a partition $\mathcal{P}$ on a nonempty set X denoted by

$$P_E(X, \mathcal{P}) = \{\alpha \in P(X) \mid \forall (x, y) \in E, (x\alpha, y\alpha) \in E \text{ and } X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\},$$

where $\mathcal{P} = \{X_i \mid i \in \Lambda\}$, a family of nonempty subsets of X satisfying $X = \bigcup_{i\in\Lambda} X_i$ and $X_i \cap X_j = \emptyset$ for all $i, j \in \Lambda$ such that $i \neq j$.

2. **Preliminaries.** In this section, we would like to introduce the reader to the basic concepts of the abstract theory of semigroups.

A *binary operation* on a set S is a mapping from the set of all order pairs of elements of S into S.

A binary operation $\cdot$ on a set S is called *associative* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in S$.

A *semigroup* is a system $(S, \cdot)$ consisting of a nonempty set S together with the binary associative operation $\cdot$, i.e., $a \cdot b$ belongs to S and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all elements a, b, c in S.

For convenience, we write S instead of $(S, \cdot)$ and let ab stand for $a \cdot b$ for any elements a, b in S.

A subset T of a semigroup S is called a *subsemigroup* of S if T is a semigroup under the operation of S.

A non-empty set T of a semigroup S is a subsemigroup of S if $a, b \in T$, then $ab \in T$.

The intersection of any set of subsemigroup of S is either an empty set or a subsemigroup of S.

An element a of a semigroup S is called a *left (right) magnifying element* if there exists a proper subset M of S such that $aM = S$ ($Ma = S$).

Let X be a nonempty set. A partial transformation of X is the collection of mappings from a subset of X into X with composition which is denoted by $P(X)$. Throughout the rest of this paper, for any $\alpha, \beta \in P(X)$ and $x \in X$, the notations $x\alpha$ and $x\alpha\beta$ are used instead of $\alpha(x)$ and $(\beta \circ \alpha)(x)$, respectively. By the composition of mappings, it is closed and associative law holds. Therefore, $P(X)$ is a semigroup under the composition of functions. We then call $P(X)$ the *partial transformation semigroup*.

The transformation on a nonempty set X is a mapping of X into itself. We denote $T(X)$ the set of all transformation on X. Clearly, $T(X)$ is a subsemigroup of $P(X)$. We then call $T(X)$ *the full transformation semigroup*.

A mapping α from a set X into itself is said to be *surjective* if every element of $x' \in X$ there exists a least one element $x \in X$ such that $x\alpha = x'$.

A mapping α from a set X into itself is said to be *injective* if distinct elements of X are mapped by α into distinct elements, i.e., if $x_1\alpha = x_2\alpha$, then $x_1 = x_2$.

A *monoid* is a semigroup S containing an identity element $e \in S$ such that for all $a \in S$, $ea = a = ae$.

Since the identity mapping on X belongs to $P(X)$, we have $P(X)$ as a monoid.

If X is a non-empty set, then a subset E of the direct product $X \times X$ is called a *relation* on X. For elements $x, y \in X$, we may write $(x, y) \in E$ or xEy if x relates to y by a relation E. A relation E on a set X is called an *equivalence relation* on X if it satisfies the following properties:

1) reflexive, i.e., $(x, x) \in E$ for all $x \in X$,
2) symmetric, i.e., for all $x, y \in X$, if $(x, y) \in E$, then $(y, x) \in E$,
3) transitive, i.e., for all $x, y, z \in X$, if $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$.

In this paper, the equivalence class of element x in a nonempty set X determined by E is denoted by $[x]_E = \{y \in X \mid xEy\}$. Let $X/E = \{[x]_E \mid x \in X\}$ and $(X_i, x_j) = X_i \cap [x_j]_E$ for $x_j \in X$ and $X_i \in \mathcal{P}$.

Let $P_E(X) = \{\alpha \in P(X) \mid \forall(x, y) \in E, (x\alpha, y\alpha) \in E\}$ denote the set of all partial transformations on X that preserves an equivalence relation E.

Theorem 2.1. *$P_E(X)$ is a submonoid of $P(X)$.*

Proof: Clearly, $P_E(X)$ is a subset of $P(X)$ and the identity mapping on X belongs to $P_E(X)$. Let $\alpha, \beta \in P_E(X)$ and let $x, y \in X$ be such that $(x, y) \in E$. Then $(x\alpha, y\alpha) \in E$ and hence $(x\alpha\beta, y\alpha\beta) \in E$. So $\alpha\beta \in P_E(X)$. Therefore, $P_E(X)$ is a submonoid of $P(X)$. □

Theorem 2.2. [15] *Let E be an equivalence relation on a set X. A function $\alpha \in P_E(X)$ is a left magnifying element if and only if α is injective but not surjective,* $\operatorname{dom}\alpha = X$ *and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Theorem 2.3. [15] *Let E be an equivalence relation on a set X. A function $\alpha \in P_E(X)$ is a right magnifying element if and only if α is surjective, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$ and either*

1) $\operatorname{dom}\alpha \neq X$ *or*
2) $\operatorname{dom}\alpha = X$ *and α is not injective.*

Let $P(X, \mathcal{P}) = \{X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\}$ denote the set of all partial transformations on X that preserves a partition $\mathcal{P}$ on X.

Theorem 2.4. *$P(X, \mathcal{P})$ is a submonoid of $P(X)$.*

Proof: Clearly, $P(X, \mathcal{P})$ is a subset of $P(X)$ and the identity mapping on X belongs to $P(X, \mathcal{P})$. Let $\alpha, \beta \in P(X, \mathcal{P})$ and let $x \in X_i$ for some $i \in \Lambda$. Then $x\alpha \in X_i$ and hence $x\alpha\beta \in X_i$. So $\alpha\beta \in P(X, \mathcal{P})$. Therefore, $P(X, \mathcal{P})$ is a submonoid of $P(X)$. □

We then define

$$P_E(X, \mathcal{P}) = \{\alpha \in P(X) \mid \forall (x, y) \in E, (x\alpha, y\alpha) \in E \text{ and } X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\},$$

which is an intersection of $P_E(X)$ and $P(X, \mathcal{P})$. Since the identity mapping on X belongs to $P_E(X)$ and $P(X, \mathcal{P})$, $P_E(X, \mathcal{P})$ is non-empty and hence $P_E(X, \mathcal{P})$ is a subsemigroup of $P_E(X)$ and $P(X, \mathcal{P})$. Evidently, $P_E(X, \mathcal{P})$ is a submonoid of $P(X)$.

Note that if the equivalence relation E is trivial, i.e., $E = X \times X$ or $E = id_X$, then $P_E(X, \mathcal{P}) = P(X, \mathcal{P})$; and if $\mathcal{P} = \{X\}$, then $P_E(X, \mathcal{P}) = P_E(X)$. Moreover, $P_E(X, \mathcal{P}) = P(X)$ if the equivalence relation E is trivial and the partition $\mathcal{P} = \{X\}$. If all elements in $\mathcal{P}$ are singleton sets, then $P_E(X, \mathcal{P})$ is the set of all restrictions of the identity function on a set X to a subset A of X.

3. Main Results.

3.1. Left magnifying elements in $P_E(X, \mathcal{P})$.

In this section, we provide the existence and some properties of left magnifying elements in $P_E(X, \mathcal{P})$. The necessary and sufficient conditions of functions in this monoid to be left magnifying elements are established.

Lemma 3.1. *If α is a left magnifying element in $P_E(X, \mathcal{P})$, then α is an injection and* $\operatorname{dom}\alpha = X$.

Proof: Assume that α is a left magnifying element in $P_E(X, \mathcal{P})$. Then there is a proper subset M of $P_E(X, \mathcal{P})$ such that $\alpha M = P_E(X, \mathcal{P})$. Since the identity map id_X on X belongs to $P_E(X, \mathcal{P})$, $\alpha\beta = id_X$ for some $\beta \in M$. This implies that α is injective and $\operatorname{dom}\alpha = X$. □

Nevertheless, the converse of Lemma 3.1 is not true in general since there is no proper subset M of $P_E(X, \mathcal{P})$ such that $id_X M = P_E(X, \mathcal{P})$.

Lemma 3.2. *Let α be a left magnifying element in $P_E(X, \mathcal{P})$. For any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Proof: Assume that α is a left magnifying element in $P_E(X,\mathcal{P})$. Then there is a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$. Then $\alpha\beta = id_X$ for some $\beta \in M$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. Therefore, $(x, y) = (xid_X, yid_X) = (x\alpha\beta, y\alpha\beta) \in E$ since $\beta \in P_E(X,\mathcal{P})$. $\square$

Note that for any function $\alpha \in P_E(X,\mathcal{P})$, for any $j \in \Lambda$, $x\alpha \in X_j$ implies $x \in X_j$ as well.

Lemma 3.3. *If $\alpha \in P_E(X,\mathcal{P})$ is bijective on X, then α is not a left magnifying element.*

Proof: Assume that $\alpha \in P_E(X,\mathcal{P})$ is bijective on X. So α^{-1} is also bijective on X. Suppose that α is a left magnifying element. Then there exists a proper subset M of $P_E(X,\mathcal{P})$ satisfying $\alpha M = P_E(X,\mathcal{P})$. Clearly, $\alpha M = \alpha P_E(X,\mathcal{P})$. Therefore, $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha P_E(X,\mathcal{P}) = P_E(X,\mathcal{P})$, which is a contradiction. $\square$

By Lemmas 3.1, 3.2, and 3.3, we obtain the following corollary.

Corollary 3.1. *If α is a left magnifying element in $P_E(X,\mathcal{P})$, then α is injective but not surjective,* $\operatorname{dom}\alpha = X$ *and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Lemma 3.4. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If $X_i \in \mathcal{P}$ is finite for all $i \in \Lambda$, then there exists no left magnifying element in $P_E(X,\mathcal{P})$.*

Proof: Suppose to the contrary that there is a left magnifying element α in $P_E(X,\mathcal{P})$. By assumption and Lemma 3.1, $\alpha|_{X_i}$ is bijective for all $i \in \Lambda$. Since $X\alpha = \left(\bigcup_{i\in\Lambda} X_i\right)\alpha = \bigcup_{i\in\Lambda} X_i\alpha = \bigcup_{i\in\Lambda} X_i = X$, α is surjective which is a contradiction. $\square$

From Lemma 3.4, it is noticeable that if a left magnifying element exists in $P_E(X,\mathcal{P})$, then $X_i \in \mathcal{P}$ is infinite for some $i \in \Lambda$. Nevertheless, the converse of this statement is not true in general. It is illustrated by the following counterexample.

Example 3.1. *Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N} \cup \{0\}\}$ be a partition on X where $X_0 = \{-(2n-1), 2n-1 \mid n \in \mathbb{N}\} \cup \{0\}$ and $X_i = \{-2i, 2i\}$ for all $i \in \mathbb{N}$. Define a relation E on X by $E = \bigcup_{j=1}^{\infty}(A_j \times A_j)$, where $A_1 = \{0, \pm1, \pm2\}$ and $A_j = \{\pm(2j-1), \pm2j\}$ for all positive integers $j \geq 2$. Clearly, $X_0 \in \mathcal{P}$ is infinite and E is an equivalence relation on X. We can see that every injection on X in $P_E(X,\mathcal{P})$ is surjective on X. Then there exists no left magnifying element in $P_E(X,\mathcal{P})$.*

As an immediate consequence of Lemma 3.4 we have the following result.

Corollary 3.2. *If X is a finite set, then $P_E(X,\mathcal{P})$ has no left magnifying elements.*

Lemma 3.5. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in P_E(X,\mathcal{P})$ is injective but not surjective,* $\operatorname{dom}\alpha = X$ *and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$, then α is a left magnifying element.*

Proof: Assume that $\alpha \in P_E(X,\mathcal{P})$ is injective but not surjective, $\operatorname{dom}\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \operatorname{dom}\beta \subseteq \operatorname{ran}\alpha\}$. Clearly, M is a proper subset of $P_E(X,\mathcal{P})$. Claim that $\alpha M = P_E(X,\mathcal{P})$. Let $\gamma \in P_E(X,\mathcal{P})$. Choose $y_x \in \operatorname{dom}\alpha$ such that $y_x\alpha = x$ for each $x \in (\operatorname{dom}\gamma)\alpha$. Define a function $\beta \in P(X)$ by $x\beta = y_x\gamma$ for all $x \in (\operatorname{dom}\gamma)\alpha$. To show that $\beta \in M$, let $a, b \in (\operatorname{dom}\gamma)\alpha$ be such that $(a, b) \in E$. By Lemma 3.2, there exist $y_a, y_b \in \operatorname{dom}\alpha$ such that $y_a\alpha = a$, $y_b\alpha = b$ and $(y_a, y_b) \in E$. Therefore, $(a\beta, b\beta) = (y_a\gamma, y_b\gamma) \in E$ since $\gamma \in P_E(X,\mathcal{P})$. Next, let $x \in (\operatorname{dom}\gamma)\alpha$ be such that $x \in X_i$ for some $X_i \in \mathcal{P}$. Then there exists $y_x \in \operatorname{dom}\alpha$ such that $y_x\alpha = x$ and $y_x \in X_i$. Therefore, $x\beta = y_x\gamma \in X_i$ since

$\gamma \in P_E(X, \mathcal{P})$. Therefore, $\beta \in P_E(X, \mathcal{P})$. Clearly, $\operatorname{dom}\beta = (\operatorname{dom}\gamma)\alpha \subseteq \operatorname{ran}\alpha$. Therefore, $\beta \in M$. Let $x \in \operatorname{dom}\gamma$. Then $x\alpha\beta = y_{x\alpha}\gamma$. Since $y_{x\alpha}\alpha = x\alpha$ and α is injective, $y_{x\alpha} = x$. Hence, $x\alpha\beta = x\gamma$. □

The following examples illuminate the ideas of the proof given in Lemma 3.5.

Example 3.2. *Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6, 7, 8, 9, \ldots\}\}$ be a partition on X. Define a relation E on X by $(x, y) \in E$ if and only if $\lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor$. It is obvious that E is an equivalence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{9, 10, 11\}, \ldots\}$. We now see that $\{6, 7, 8, \ldots\} \in \mathcal{P}$ is infinite. Let α be a function defined by*

$$x\alpha = \begin{cases} x & \text{if } x \leq 5, \\ x + 3 & \text{if } x > 5. \end{cases}$$

It is easy to see that $\alpha \in P_E(X, \mathcal{P})$ is injective but not surjective, $\operatorname{dom}\alpha = X$ *and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Lemma 3.5, α is a left magnifying element. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \operatorname{dom}\beta \subseteq \operatorname{ran}\alpha\}$ and consider the element $\gamma \in P_E(X, \mathcal{P})$, which is defined by*

$$x\gamma = \begin{cases} x & \text{if } x \leq 4, \\ x - 3 & \text{if } x > 8. \end{cases}$$

Then there exists an element $\beta \in M$ such that $\alpha\beta = \gamma$. We illustrate the idea by considering $9, 10 \in \operatorname{dom}\gamma$. Hence, $12, 13 \in (\operatorname{dom}\gamma)\alpha$ such that $y_{12} = 9$ and $y_{13} = 10$. Therefore, $12\beta = y_{12}\gamma = 9\gamma = 6$ and $13\beta = y_{13}\gamma = 10\gamma = 7$. To get the desired result, define a function β in $P_E(X, \mathcal{P})$ by

$$x\beta = \begin{cases} x & \text{if } x \leq 4, \\ x - 6 & \text{if } x > 11. \end{cases}$$

Clearly, $\beta \in M$ and $\alpha\beta = \gamma$.

By Corollary 3.1 and Lemma 3.5, we obtain the following theorem.

Theorem 3.1. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. A function $\alpha \in P_E(X, \mathcal{P})$ is a left magnifying element if and only if α is injective but not surjective,* $\operatorname{dom}\alpha = X$ *and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Although Theorem 3.1 resembles our results in [15], the construction of the proof is more complicated because it is influenced by preserving both an equivalence relation and a partition. Even we obtained the result that if $\mathcal{P} = \{X_i \mid i \in \Lambda\}$, a partition on a set X having at least one infinite partition, then we can find the conditions for elements to be a left magnifying element, but we may fail to find a such element satisfying those conditions. However, in case there is exactly one element $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$ for all $x \in X$, the existence of left magnifying elements is proved in the next theorem.

Theorem 3.2. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that for each $x \in X$, there is exactly one $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$. There exists a left magnifying element in $P_E(X, \mathcal{P})$ if and only if there is $X_j \in \mathcal{P}$ such that X_j is infinite.*

Proof: The necessity is obtained by Lemma 3.4. Conversely, suppose that there exists $X_j \in \mathcal{P}$ such that X_j is infinite.

Case 1: There exists $t \in X$ such that (X_j, t) is infinite. Then there is a proper subset S of (X_j, t) such that $|S| = |(X_j, t)| = |(X_j, t) \setminus S|$. So there is a bijection γ from (X_j, t)

to S. Define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in (X_j, t), \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X, \mathcal{P})$ and α is injective. Hence, $\operatorname{ran}\alpha \subseteq X \setminus ((X_j, t) \setminus S) \neq X$. Then α is injective but not surjective. Obviously, $\operatorname{dom}\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.1, α is a left magnifying element.

Case 2: (X_j, t) is finite for all $t \in X$.

Case 2.1: There is a natural number n such that $K = \{(X_j, t) \mid t \in X \text{ and } |(X_j, t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijection λ from K to K'. So $|A| = |A\lambda| = n$ for all $A \in K$. Hence, for all $A \in K$, there exists a bijection η_A from A to $A\lambda$. Let $\eta = \bigcup\limits_{A \in K} \eta_A$. Then η is a bijection from $\bigcup\limits_{A \in K} A$ to $\bigcup\limits_{A \in K'} A$. Define a function α by

$$x\alpha = \begin{cases} x\eta & \text{if } x \in \bigcup\limits_{A \in K} A, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X, \mathcal{P})$ is injective. Since $\operatorname{ran}\alpha = X \setminus \bigcup\limits_{A \in K \setminus K'} A \neq X$, α is not surjective. Obviously, $\operatorname{dom}\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.1, α is a left magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_j, t) \mid t \in X \text{ and } |(X_j, t)| = n\}$ is finite. Then for each $t \in X$, there exists $t' \in X$ such that $|(X_j, t)| < |(X_j, t')|$. Let $A = \{(X_j, t) \mid [t]_E \subseteq X_j\}$. In this case, A is an infinite set. Let $n_1 = \min\limits_{(X_j, t) \in A} |(X_j, t)|$ and $K_1 = \{(X_j, t) \mid |(X_j, t)| = n_1\}$. Choose $(X_j, t_1) \in K_1$. Let $n_2 = \min\limits_{(X_j, t) \in A_1} |(X_j, t)|$ where $A_1 = A \setminus K_1$ and $K_2 = \{(X_j, t) \mid |(X_j, t)| = n_2\}$. Choose $(X_j, t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_j, t_1), (X_j, t_2), \ldots, (X_j, t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min\limits_{(X_j, t) \in A_k} |(X_j, t)|$ where $A_k = A \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_j, t_k) \in K_k$, where $K_k = \{(X_j, t) \mid |(X_j, t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < \cdots < n_k < \cdots$. Next, we let $B = \{(X_j, t_l) \mid l \geq 1\}$. Then $|(X_j, t_l)| < |(X_j, t_{l+1})|$ for all $l \geq 1$. Hence, there exists an injection $\gamma_l : (X_j, t_l) \to (X_j, t_{l+1})$. Let $\gamma = \bigcup\limits_{l \geq 1} \gamma_l$. Then γ is an injection on $\bigcup\limits_{C \in B} C$. Next, define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup\limits_{C \in B} C, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X, \mathcal{P})$ and α is injective. Since $\operatorname{ran}\alpha \subseteq X \setminus (X_j, t_1) \neq X$, α is not surjective. Obviously, $\operatorname{dom}\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.1, α is a left magnifying element. $\square$

3.2. **Right magnifying elements in $\boldsymbol{P_E(X, \mathcal{P})}$.** In this section, we provide the existence and some properties of right magnifying elements in $P_E(X, \mathcal{P})$. The necessary and sufficient conditions of functions in this monoid to be right magnifying elements are established.

Lemma 3.6. *If α is a right magnifying element in $P_E(X,\mathcal{P})$, then α is surjective.*

Proof: Assume that α is a right magnifying element in $P_E(X,\mathcal{P})$. Then there exists a proper subset M of $P_E(X,\mathcal{P})$ satisfying $M\alpha = P_E(X,\mathcal{P})$. Clearly, the identity map id_X on X belongs to $P_E(X,\mathcal{P})$. So there exists $\beta \in M$ such that $\beta\alpha = id_X$. This implies that α is surjective. □

It is clear that for any surjection $\alpha \in P_E(X,\mathcal{P})$, if $x \in X_i$, then there exists an element $a \in X_i$ such that $a\alpha = x$. Consequently, any right magnifying element has this property by Lemma 3.6.

Lemma 3.7. *Let α be a right magnifying element in $P_E(X,\mathcal{P})$. For any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$, $y = b\alpha$.*

Proof: Assume that α is a right magnifying element in $P_E(X,\mathcal{P})$. Then there exists a proper subset M of $P_E(X,\mathcal{P})$ satisfying $M\alpha = P_E(X,\mathcal{P})$. Since $id_X \in P_E(X,\mathcal{P})$, $\beta\alpha = id_X$ for some $\beta \in M$. Let $x,y \in X$ be such that $(x,y) \in E$. Then $x\beta\alpha = x$ and $y\beta\alpha = y$. Since $\beta \in P_E(X,\mathcal{P})$, we have $(x\beta, y\beta) \in E$. Choose $a = x\beta$ and $b = y\beta$. This completes the proof. □

Lemma 3.8. *If $\alpha \in P_E(X,\mathcal{P})$ is bijective and* $\operatorname{dom}\alpha = X$, *then α is not a right magnifying element.*

Proof: Assume that $\alpha \in P_E(X,\mathcal{P})$ is bijective and $\operatorname{dom}\alpha = X$. So α^{-1} is bijective on X. Suppose that α is a right magnifying element. Then there is a proper subset M of $P_E(X,\mathcal{P})$ satisfying $M\alpha = P_E(X,\mathcal{P})$. Clearly, $M\alpha = P_E(X,\mathcal{P})\alpha$. Therefore, $M = M\alpha\alpha^{-1} = P_E(X,\mathcal{P})\alpha\alpha^{-1} = P_E(X,\mathcal{P})$, which is a contradiction. □

By Lemmas 3.6, 3.7, and 3.8, we obtain the following corollary.

Corollary 3.3. *If α is a right magnifying element in $P_E(X,\mathcal{P})$ and* $\operatorname{dom}\alpha = X$, *then α is surjective but not injective and for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$, $y = b\alpha$.*

Lemma 3.9. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If X_i is finite for all $i \in \Lambda$, then there exists no right magnifying element in $P_E(X,\mathcal{P})$.*

Proof: Suppose to the contrary that there is a right magnifying element $\alpha \in P_E(X,\mathcal{P})$. By assumption and Lemma 3.6, α is surjective and hence $\operatorname{dom}\alpha = X$ since $X_i\alpha \subseteq X_i$ is finite for all $i \in \Lambda$. So $\alpha|_{X_i}$ is surjective on X_i and hence $\alpha|_{X_i}$ is bijective on X_i. Since $X\alpha = \left(\bigcup_{i\in\Lambda} X_i\right)\alpha = \bigcup_{i\in\Lambda} X_i\alpha = X$, α is injective on X which is a contradiction. □

From Lemma 3.9, it is noticeable that if a right magnifying element exists in $P_E(X,\mathcal{P})$, then X_i is infinite for some $i \in \Lambda$. However, the converse of this statement is not true in general. It is illustrated by the following counterexample.

Example 3.3. *Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N} \cup \{0\}\}$ be a partition on X where $X_0 = \{0,-1,-2,\ldots\}$ and $X_i = \{2i-1, 2i\}$ for all $i \in \mathbb{N}$. Define a relation E on X by $E = \bigcup_{j=1}^{\infty}(A_j \times A_j)$ where $A_1 = \{0,\pm1,\pm2\}$ and $A_j = \{\pm(2j-1), \pm2j\}$ for all positive integers $j \geq 2$. It is easy to check that every surjection in $P_E(X,\mathcal{P})$ is bijective on X. Hence, there exists no right magnifying element in $P_E(X,\mathcal{P})$.*

Corollary 3.4. *If X is a finite set, then $P_E(X,\mathcal{P})$ has no right magnifying elements.*

Lemma 3.10. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in P_E(X,\mathcal{P})$ is surjective but not injective,* $\operatorname{dom}\alpha = X$ *and for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.*

Proof: Assume that $\alpha \in P_E(X, \mathcal{P})$ is surjective but not injective, $\operatorname{dom} \alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta$ is not surjective$\}$ and γ be a function in $P_E(X, \mathcal{P})$. Since α is surjective, for each $x \in \operatorname{dom} \gamma$ such that $x\gamma \in X_i$ for some $X_i \in \mathcal{P}$, there exists $y_x \in X_i$ such that $y_x\alpha = x\gamma$. Define $\beta \in P(X)$ by $x\beta = y_x$ for all $x \in \operatorname{dom} \gamma$ (if $x_1\gamma = x_2\gamma$, then choose $y_{x_1} = y_{x_2}$ and if $(x_1\gamma, x_2\gamma) \in E$, then choose $(y_{x_1}, y_{x_2}) \in E$). To show that $\beta \in P_E(X, \mathcal{P})$, let $a, b \in X$ be such that $(a, b) \in E$. Since γ belongs to $P_E(X, \mathcal{P})$, $(a\gamma, b\gamma) \in E$. By assumption, we can choose $(y_a, y_b) \in E$ such that $y_a\alpha = a$ and $y_b\alpha = b$. Then $(a\beta, b\beta) = (y_a, y_b) \in E$. Clearly, $\beta \in P(X, \mathcal{P})$. Therefore, $\beta \in P_E(X, \mathcal{P})$. By assumption, there exist $x, y \in X$ such that $x\alpha = y\alpha$. Thus, at least one of x and y does not belong to $\operatorname{ran} \beta$. So $\beta \in M$. For all $x \in X$, $x\beta\alpha = y_x\alpha = x\gamma$. Therefore, α is a right magnifying element. $\square$

Example 3.4. *Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}\}$ be a partition on X. Define a relation E on X by $(x, y) \in E$ if and only if $\left\lfloor \frac{x}{3} \right\rfloor = \left\lfloor \frac{y}{3} \right\rfloor$. It is obvious that E is an equivalence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{9, 10, 11\}, \ldots\}$. We now see that $\{1, 3, 5, \ldots\} \in \mathcal{P}$ is infinite. Let α be a function defined by*

$$x\alpha = \begin{cases} x & \text{if } x \leq 8, \\ x - 6 & \text{if } x > 8. \end{cases}$$

It is easy to see that $\alpha \in P_E(X, \mathcal{P})$ is surjective but not injective, $\operatorname{dom} \alpha = X$ *and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 3.10, α is a right magnifying element. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta$ is not surjective$\}$ and consider the element $\gamma \in P_E(X, \mathcal{P})$, which is defined by*

$$x\gamma = \begin{cases} x & \text{if } x \leq 5, \\ x - 12 & \text{if } x \geq 15. \end{cases}$$

Then there exists an element $\beta \in M$ such that $\beta\alpha = \gamma$. We illustrate the idea by considering $3, 4, 15, 16 \in \operatorname{dom} \gamma$. We can see that $3\gamma = 15\gamma = 3$ and $4\gamma = 16\gamma = 4$. Now we have 2 possibilities for each y_3 and y_4, i.e., $y_3 = 3$ or 9 and $y_4 = 4$ or 10. If we follow the proof of Lemma 3.10, then we choose $y_3 = y_{15} = 9$. Since $(3\gamma, 4\gamma) \in E$, we must choose $y_4 = y_{16} = 10$. To get the desired result, define a function β in $P_E(X, \mathcal{P})$ by $3\beta = 15\beta = 9$, $4\beta = 16\beta = 10$, $5\beta = 17\beta = 11$ and

$$x\beta = \begin{cases} x & \text{if } x \leq 2, \\ x - 6 & \text{if } x \geq 18. \end{cases}$$

Clearly, $\beta \in M$ and $\beta\alpha = \gamma$.

Lemma 3.11. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$ and $T_E(X, \mathcal{P}) = \{\alpha \in P_E(X, \mathcal{P}) \mid \operatorname{dom} \alpha = X\}$. If $\alpha \in P_E(X, \mathcal{P}) \setminus T_E(X, \mathcal{P})$ is surjective and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.*

Proof: Let $\alpha \in P_E(X, \mathcal{P}) \setminus T_E(X, \mathcal{P})$ be surjective and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta$ is not surjective$\}$. To show that $M\alpha = P_E(X, \mathcal{P})$, let γ be a function in $P_E(X, \mathcal{P})$. Since α is surjective, for each $x\gamma \in \operatorname{ran} \gamma$, we can choose $y_{x\gamma} \in X$ such that $y_{x\gamma}\alpha = x\gamma$. Then we define a function β by $x\beta = y_{x\gamma}$ for all $x \in \operatorname{dom} \gamma$. Clearly, $\beta \in P(X)$. Let $a, b \in \operatorname{dom} \gamma$ be such that $(a, b) \in E$. Then $(a\gamma, b\gamma) \in E$ since $\gamma \in P_E(X, \mathcal{P})$. By assumption, there exists $(a\beta, b\beta) = (y_{a\gamma}, y_{b\gamma}) \in E$ such that $y_{a\gamma}\alpha = a\gamma$ and $y_{b\gamma}\alpha = b\gamma$. Let $a \in \operatorname{dom} \gamma$ be such that $a \in X_i$. Hence, $a\gamma \in X_i$. Then there exists $y_{a\gamma} \in X_i$ such that $y_{a\gamma}\alpha = a\gamma$. So

$a\beta = y_{a\gamma} \in X_i$. Since $\operatorname{ran}\beta \subseteq \operatorname{dom}\alpha \neq X$, β is not surjective. Thus, $\beta \in M$. For all $x \in X$, $x\beta\alpha = y_{x\gamma}\alpha = x\gamma$. Therefore, α is a right magnifying element. $\square$

Example 3.5. *Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1,3,5,\ldots\},\{2,4,6,\ldots\}\}$ be a partition on X. Define a relation E on X by $(x,y) \in E$ if and only if $\lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor$. It is obvious that E is an equivalence relation on X and $X/E = \{\{1,2\},\{3,4,5\},\{6,7,8\},\{9,10,11\},\ldots\}$. We now see that $\{1,3,5,\ldots\} \in \mathcal{P}$ is infinite. Let α be a function defined by*

$$x\alpha = \begin{cases} x & \text{if } x \leq 5, \\ x-6 & \text{if } x \geq 9. \end{cases}$$

It is easy to see that $\alpha \in P_E(X,\mathcal{P})$ is surjective, $\operatorname{dom}\alpha \neq X$, and for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \beta$ is not surjective$\}$ and consider the element $\gamma \in P_E(X,\mathcal{P})$, which is defined by $x\gamma = x$ for all odd positive integers. By Lemma 3.11, α is a right magnifying element and there exists an element $\beta \in M$ such that $\beta\alpha = \gamma$. To get the desired result, define a function β in $P_E(X,\mathcal{P})$ by $1\beta = 1$ and $x\beta = x+6$ for all odd positive integers $x \geq 3$. Clearly, $\beta \in M$ and $\beta\alpha = \gamma$.

The next example shows that α is a right magnifying element such that $\operatorname{dom}\alpha \neq X$ and α is bijective.

Example 3.6. *Let $X = \mathbb{N}$ and $\mathcal{P} = \{X_1, X_2\}$ be a partition on X where $X_1 = \{1,3,5,\ldots\}$ and $X_2 = \{2,4,6,\ldots\}$. Define a relation E on X by $(x,y) \in E$ if and only if $\lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor$. It is obvious that E is an equivalence relation on X and $X/E = \{\{1,2\},\{3,4,5\},\{6,7,8\},\{9,10,11\},\ldots\}$. We now see that $X_1 \in \mathcal{P}$ is infinite. Let α be a function defined by $3\alpha = 1$, $4\alpha = 2$ and*

$$x\alpha = \begin{cases} x & \text{if } |(X_1,x)| = 1, \\ x-6 & \text{if } |(X_1,x)| = 2, \end{cases}$$

for all $x > 5$. It is easy to see that $\alpha \in P_E(X,\mathcal{P})$ is bijective, $\operatorname{dom}\alpha \neq X$, and for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \beta$ is not surjective$\}$ and consider the element $\gamma \in P_E(X,\mathcal{P})$, which is defined by $x\gamma = x$ if $x \leq 5$ and $x\gamma = x-6$ if $x \geq 12$. By Lemma 3.11, α is a right magnifying element and there exists an element $\beta \in M$ such that $\beta\alpha = \gamma$. To get the desired result, define a function β in $P_E(X,\mathcal{P})$ by $x\beta = x+2$ if $x = 1,2$, $x\beta = x+6$ if $x = 3,4,5$ and

$$x\beta = \begin{cases} x & \text{if } |(X_1,x)| = 2, \\ x-6 & \text{if } |(X_1,x)| = 1, \end{cases}$$

for all $x > 11$. Clearly, $\beta \in M$ and $\beta\alpha = \gamma$.

By Lemmas 3.8 and 3.11 and Corollary 3.3, we obtain the following theorem.

Theorem 3.3. *Let E be an equivalence relation on a set X and $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on X such that X_i is infinite for some $i \in \Lambda$. A function $\alpha \in P_E(X,\mathcal{P})$ is a right magnifying element if and only if α is surjective, for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$ and either*

1) $\operatorname{dom}\alpha \neq X$ *or*

2) $\operatorname{dom}\alpha = X$ *and α is not injective.*

Although Theorem 3.3 resembles our results in [15], the construction of the proof is more complicated because it is influenced by preserving both an equivalence relation and a partition. Even we obtained the result that if $\mathcal{P} = \{X_i \mid i \in \Lambda\}$, a partition on a set X having at least one infinite partition, then we can find the conditions for elements to be a right magnifying element, but we may fail to find a such element satisfying those conditions. However, in case there is exactly one element $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$ for all $x \in X$, the existence of right magnifying elements is proved in the next theorem.

Theorem 3.4. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that for each $x \in X$, there is exactly one $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$. There exists a right magnifying element in $P_E(X, \mathcal{P})$ if and only if there is $X_j \in \mathcal{P}$ such that X_j is infinite.*

Proof: The necessity is obtained by Lemma 3.9. On the other hand, suppose that there exists $X_j \in \mathcal{P}$ such that X_j is infinite.

Case 1: There exists $t \in X$ such that (X_j, t) is infinite. Then there is a proper subset A of (X_j, t) such that $|A| = |(X_j, t)| = |(X_j, t) \setminus A|$. So there is a bijective function γ from A to (X_j, t). Define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in A, \\ x & \text{if } x \in X \setminus (X_j, t). \end{cases}$$

Clearly, $\operatorname{dom}\alpha \neq X$, α is surjective and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.3, α is a right magnifying element.

Case 2: (X_j, t) is finite for all $t \in X$.

Case 2.1: There is a natural number n such that $K = \{(X_j, t) \mid t \in X \text{ and } |(X_j, t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijective function λ from K' to K. So $|A| = |A\lambda| = n$ for all $A \in K'$. Hence, for all $A \in K'$, there exists a bijective function γ_A from A to $A\lambda$. Let $\gamma = \bigcup\limits_{A \in K'} \gamma_A$. Then γ is a bijection from $\bigcup\limits_{A \in K'} A$ to $\bigcup\limits_{A \in K} A$. Define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup\limits_{A \in K'} A, \\ x & \text{if } x \notin \bigcup\limits_{A \in K} A. \end{cases}$$

Clearly, $\operatorname{dom}\alpha \neq X$ and α is surjective and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.3, α is a right magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_j, t) \mid t \in X \text{ and } |(X_j, t)| = n\}$ is finite. Then for each $t \in X$, there exists $t' \in X$ such that $|(X_j, t)| < |(X_j, t')|$. Let $\mathcal{C} = \{(X_j, t) \mid [t]_E \subseteq X_j\}$. In this case, $\mathcal{C}$ is an infinite set. Let $n_1 = \min\limits_{(X_j,t) \in \mathcal{C}} |(X_j, t)|$ and $K_1 = \{(X_j, t) \mid |(X_j, t)| = n_1\}$. Choose $(X_j, t_1) \in K_1$. Let $n_2 = \min\limits_{(X_j,t) \in \mathcal{C}_1} |(X_j, t)|$ where $\mathcal{C}_1 = \mathcal{C} \setminus K_1$ and $K_2 = \{(X_j, t) \mid |(X_j, t)| = n_2\}$. Choose $(X_j, t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_j, t_1), (X_j, t_2), \ldots, (X_j, t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min\limits_{(X_j,t) \in \mathcal{C}_k} |(X_j, t)|$ where $\mathcal{C}_k = \mathcal{C} \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_i, t_k) \in K_k$, where $K_k = \{(X_j, t) \mid |(X_j, t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < \cdots < n_k < \cdots$. Next, we let $A = \{(X_j, t_i) \mid i \geq 1\}$. Then $|(X_j, t_i)| < |(X_j, t_{i+1})|$ for all $i \geq 1$. Hence, there exists a surjection $\gamma_i : (X_j, t_i) \to (X_j, t_{i-1})$ for all $i \geq 2$. Let $\gamma = \bigcup\limits_{i \geq 2} \gamma_i$. Then γ is a surjection from $\bigcup\limits_{B \in A} B \setminus (X_j, t_1)$ to $\bigcup\limits_{B \in A} B$. Next, define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup_{B \in A} B \setminus (X_j, t_1), \\ x & \text{if } x \in X \setminus \bigcup_{B \in A} B. \end{cases}$$

Clearly, $\operatorname{dom}\alpha \neq X$, α is surjective and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.3, α is a right magnifying element. □

4. **Conclusion.** Let E be an equivalence relation on a nonempty set X and $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on X. If X_i is finite for all $i \in \Lambda$, then neither left magnifying element nor right magnifying element exists in $P_E(X, \mathcal{P})$. Assume that X_i is infinite for some $i \in \Lambda$. Each of the following statements holds true.

(1) A function $\alpha \in P_E(X, \mathcal{P})$ is a left magnifying element if and only if α is injective but not surjective, $\operatorname{dom}\alpha = X$, and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.

(2) A function $\alpha \in P_E(X, \mathcal{P})$ is a right magnifying element if and only if α is surjective, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$ and either
(a) $\operatorname{dom}\alpha \neq X$ or
(b) $\operatorname{dom}\alpha = X$ and α is not injective.

(3) There are magnifying elements in $P_E(X, \mathcal{P})$ if and only if there is $X_i \in \mathcal{P}$ such that X_i is infinite, provided that there is exactly one $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$ for all $x \in X$.

Because of the influence of the simultaneous preserving an equivalence relation and a partition, the existence of magnifying elements in the last statement has the incremental part over the literature. It relies on the condition that there is exactly one $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$ for all $x \in X$. And above all, its ideas can be applied in the more complex problems which actually occur in the world. For example, one job needs these algorithms a_1, a_2, a_3, a_4, ... The employee must weigh each algorithm that they want to do under the following conditions.

(I) a_1 needs a leadership skill.
(II) a_2, a_3, a_4 need a writing skill.
(III) a_5, a_6, a_7, ... need an interpersonal skill.
(IV) Everyone does not need to weigh for every algorithm.
(V) If somebody want to weigh a_1 and a_2, then he must weigh a_1 as 1 and a_2 as 2.
(VI) If somebody want to weigh a_3 and a_4, then he must weigh a_2 and a_3 as 3 or 4.
(VII) Each algorithm must have only one weight. However, some algorithms can have the same weight.
(VIII) The algorithms a_5, a_6, a_7, ... must have weight over 5, and it cannot be weighed 1, 2, 3 and 4.

From this situation, we can establish the set of algorithm $X = \{a_1, a_2, a_3, \ldots\}$ corresponding to a set of natural numbers. By the conditions (I)-(III), all of algorithms a_2, a_3, a_4 use the same skill. This implies that if someone can do the algorithm a_2, then he can do the algorithm a_3. So it can be substituted for each other. Similarly, $a_5, a_6, a_7, \ldots$ also use the same skill and hence it can be substituted for each other. Thus, we can set an equivalence relation E on a set $X, E = \{\{a_1\}, \{a_2, a_3, a_4\}, \{a_5, a_6, a_7, \ldots\}\}$. Moreover, we set a partition $\mathcal{P}$ on a set X, $\mathcal{P} = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6, a_7, \ldots\}\}$. From each condition, $P_E(X, \mathcal{P})$ is formed. By the conclusion (3), there are magnifying elements in $P_E(X, \mathcal{P})$. For example, A weighs each algorithm as follows:

Algorithms	Weight	Arrangement of abilities
a_1	1	a_1
a_2	2	a_2
a_3	4	a_4
a_4	3	a_3
a_5	6	a_6
a_6	5	a_5
a_7	7	a_7
a_8	8	a_8
a_9	10	a_{10}
a_{10}	9	a_9
$\vdots$	$\vdots$	$\vdots$

If we let α be a capable function of A which transforms each algorithm into an arrangement of abilities, then α is a left magnifying element by the conclusion (1). Therefore, A has the opportunity to be appointed as a leader in case the chain of command is from the leader to the employee. Similarly, if the process of work is from employee to an approving of the leader, then we may look for the right magnifying elements. For example, B is not good at doing a_5 and a_6. So he does not weigh a_5 and a_6 as follows:

Algorithms	Weight	Arrangement of abilities
a_1	1	a_1
a_2	2	a_2
a_3	4	a_4
a_4	3	a_3
a_5	-	-
a_6	-	-
a_7	5	a_5
a_8	5	a_5
a_9	6	a_6
a_{10}	7	a_7
$\vdots$	$\vdots$	$\vdots$

If we let β be a capable function of B which transforms each algorithm into an arrangement of abilities, then β cannot be a left magnifying element since $\operatorname{dom}\alpha \neq X$. By the conclusion (2), we have β as a right magnifying element. Therefore, B has the opportunity to be appointed as a leader in this process.

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