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ON TRIPOLAR FUZZY INTERIOR IDEALS IN ORDERED SEMIGROUPS

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Abstract. *Tripolar fuzzy sets are a concept that deals with tripolar information. This idea is a generalization of bipolar and intuitionistic fuzzy sets. In this paper, we introduce the concepts of tripolar fuzzy interior ideals in ordered semigroups, we prove that in regular, intra-regular, and semisimple ordered semigroups, the concepts of tripolar fuzzy interior ideals and tripolar fuzzy ideals coincide. We characterize semisimple ordered semigroups in terms of tripolar fuzzy interior ideals. Finally, we introduce the concepts of tripolar fuzzy simple ordered semigroups. We prove that the concepts of tripolar fuzzy simple ordered semigroups and simple ordered semigroups coincide.*
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1. **Introduction.** The theory of fuzzy sets was introduced by Zadeh [1] in 1965. Fuzzy sets are the most appropriate theory for dealing with uncertainty. After the introduction of the concept of fuzzy sets by Zadeh, several researchers conducted research on the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research, and many branches of pure and applied mathematics (see [2] for example).

In semigroups, regular, intra-regular, and semisimple, the interior ideals and ideals coincide, as well in ordered semigroups. Many authors applied fuzzy theory to (ordered) semigroup theory. For example, Kehayopulu and Tsingelis [3] showed that in regular, intra-regular, and semisimple, the concepts of fuzzy interior ideals and fuzzy ideals coincide. In 2009, Khan and Shabir [4] introduced the concepts of (α, β)-fuzzy interior ideals in ordered semigroups as generalizing fuzzy interior ideals and proved that in regular, intra-regular, and semisimple ordered semigroups, the concepts of (α, β)-fuzzy interior ideals and (α, β)-fuzzy ideals coincide. Shabir and Khan [5] proved that in regular, intra-regular, and semisimple, the concepts of intuitionistic fuzzy interior ideals and intuitionistic fuzzy

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ideals coincide. Ibrar et al. [6] also proved that in regular, intra-regular, and semisimple, the (α, β)-bipolar fuzzy interior ideals and (α, β)-bipolar fuzzy ideals coincide. In 2022, Linesawat et al. [7] introduced the concepts of anti-hybrid interior ideals in ordered semigroups and showed that in regular, intra-regular, and semisimple, the concepts of anti-hybrid interior ideals and anti-hybrid ideals coincide.

The tripolar fuzzy set as a generalization of fuzzy set, bipolar fuzzy set, and intuitionistic fuzzy set, Rao [8] introduced the notion of tripolar fuzzy set and studied tripolar fuzzy interior ideals of Γ-semigroup. Rao and Venkateswarlu [9, 10] investigated tripolar fuzzy interior ideal, tripolar fuzzy soft ideal, and tripolar fuzzy soft interior ideal of Γ-semigroup and Γ-semiring. In 2020, Rao [11] introduced the notion of the tripolar fuzzy interior ideal of semigroup, tripolar fuzzy soft ideals, and tripolar fuzzy soft interior ideals over semigroup. They also studied some of their algebraic properties and the relations between them.

In this paper, some basic terminologies of ordered semigroups, fuzzy sets and tripolar fuzzy sets are provided in Section 2. We introduce the notion of tripolar fuzzy (left, right and interior) ideals in ordered semigroups in Section 3. Results in Section 3 show that the concepts of tripolar fuzzy ideals and tripolar interior ideals coincide in certain classes of ordered semigroups. Finally, we characterize simple and semisimple ordered semigroups in terms of tripolar interior ideals. Section 4 concludes the paper and indicates future work.

2. **Preliminaries.** In this section, we will recall the basic terms and definitions from the ordered semigroup theory and the hybrid structure theory that we will use later in this paper.

A *groupoid* $(S; \cdot)$ consists of a nonempty set S together with a (binary) operation $\cdot$ on S. A *semigroup* $(S; \cdot)$ is a groupoid in which the operation $\cdot$ is associative, that is, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$.

Definition 2.1 ([3])**.** *The structure $(S; \cdot, \leq)$ is called an* ordered semigroup *if the following conditions are satisfied:*
(1) $(S; \cdot)$ is a semigroup;
(2) $(S; \leq)$ is a partially ordered set;
(3) for every $a, b, c \in S$ if $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

For simplicity, we denoted an ordered semigroup $(S; \cdot, \leq)$ by its carrier set as a bold letter $\mathbf{S}$. For $K \subseteq S$, we denote

$$(K] := \{a \in S \mid a \leq k \text{ for some } k \in K\}.$$

Let A and B be two nonempty subsets of S. Then we define

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of $\mathbf{S}$ [3] if $AA \subseteq A$.

Definition 2.2 ([3])**.** *Let $\mathbf{S}$ be an ordered semigroup. A nonemty subset A of S is called a* left (resp. right) ideal *of $\mathbf{S}$ if it satisfied the following conditions:*
(1) $SA \subseteq A$ (resp. $AS \subseteq A$);
(2) for $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.

A nonempty subset I of S is called an *ideal* if it is both a left and a right ideal of $\mathbf{S}$.

Definition 2.3 ([3])**.** *Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called an* interior ideal *of $\mathbf{S}$ if it satisfied the following conditions:*

(1) $SAS \subseteq A$*;*
(2) for $x, y \in S$*, if* $x \leq y$ *and* $y \in A$*, then* $x \in A$*.*

A *fuzzy subset* of a nonempty subset X (or a *fuzzy set* in a nonempty subset X) is a mapping $f\colon X \to [0, 1]$ from X to a unit closed interval (see [1]).

Definition 2.4 ([8])**.** *A fuzzy set* A *of a universe set* X *is said to be a* tripolar fuzzy set*, if*

$$A := \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$$

where $\mu_A\colon X \to [0, 1]$*,* $\lambda_A\colon X \to [0, 1]$ *and* $\delta_A\colon X \to [-1, 0]$*. The membership degree* $\mu_A(x)$ *characterizes the extent that the element* x *satisfies the property corresponding to tripolar fuzzy set* A*,* $\lambda_A(x)$ *characterizes the extent that the element* x *satisfies the not property (irrelevant) corresponding to tripolar fuzzy set* A*, and* $\delta_A(x)$ *characterizes the extent that the element* x *satisfies the implicit counter property of tripolar fuzzy set* A*. For simplicity* $A = (\mu_A, \lambda_A, \delta_A)$ *has been used for* $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$*.*

Now, we let $Tri(S)$ be the set of all tripolar fuzzy subsets of S and define an operation on such set as follows: Let $A = (\mu_A, \lambda_A, \delta_A)$, $B = (\mu_B, \lambda_B, \delta_B)$ be elements in $Tri(S)$. Then the product $A \circ B$ of A and B as the tripolar fuzzy subset $A \circ B := (\mu_A \circ \mu_B, \lambda_A \circ \lambda_B, \delta_A \circ \delta_B)$ of S defined as follows: For each $x \in S$,

$$(\mu_A \circ \mu_B)(x) := \begin{cases} \bigvee\limits_{(a,b) \in \mathbf{S}_x} \{\min\{\mu_A(a), \mu_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda_A \circ \lambda_B)(x) := \begin{cases} \bigwedge\limits_{(a,b) \in \mathbf{S}_x} \{\max\{\lambda_A(a), \lambda_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

$$(\delta_A \circ \delta_B)(x) := \begin{cases} \bigwedge\limits_{(a,b) \in \mathbf{S}_x} \{\max\{\delta_A(a), \delta_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the structure $(Tri(S); \circ)$ is a semigroup. We define the relation $\subseteq$ on $Tri(S)$ as follows: $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$, $\lambda_A \supseteq \lambda_B$ and $\delta_A \supseteq \delta_B$ such that $\mu_A \subseteq \mu_B$, $\lambda_A \supseteq \lambda_B$ and $\delta_A \supseteq \delta_B$ mean that $\mu_A(x) \leq \mu_B(x)$, $\lambda_A(x) \geq \lambda_B(x)$ and $\delta_A(x) \geq \delta_B(x)$ for all $x \in S$, respectively. Finally, we define a binary operation $\cap$ on $Tri(S)$ as follows:

$$A \cap B := (\mu_A \cap \mu_B, \lambda_A \cup \lambda_B, \delta_A \cup \delta_B),$$

where $(\mu_A \cap \mu_B)(x) := \min\{\mu_A(x), \mu_B(x)\}$, $(\lambda_A \cup \lambda_B)(x) := \max\{\lambda_A(x), \lambda_B(x)\}$ and $(\delta_A \cup \delta_B)(x) := \max\{\delta_A(x), \delta_B(x)\}$ for all $x \in S$.

The tripolar fuzzy subset $\mathcal{S} := \left(1^+_{\mathcal{S}}, 0_{\mathcal{S}}, 1^-_{\mathcal{S}}\right)$ of S is defined by $1^+_{\mathcal{S}}(x) := 1$, $0_{\mathcal{S}}(x) := 0$ and $1^-_{\mathcal{S}}(x) := -1$ for all $x \in S$.

Let $X \subseteq S$. We denote by $\chi_X := (\mu_{\chi_X}, \lambda_{\chi_X}, \delta_{\chi_X})$ the tripolar fuzzy subset of X in S and it is defined as follows:

$$(\mu_{\chi_X})(x) := \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda_{\chi_X})(x) := \begin{cases} 0 & \text{if } x \in X, \\ 1 & \text{otherwise,} \end{cases}$$

$$(\delta_{\chi_X})(x) := \begin{cases} -1 & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

3. **Main Results.** In this section, we introduce the concepts of tripolar fuzzy interior ideals and study their algebraic properties. We prove that in regular, intra-regular, and semisimple ordered semigroups, tripolar fuzzy interior ideals and tripolar fuzzy ideals coincide. Moreover, we characterize semisimple ordered semigroups in terms of tripolar fuzzy interior ideals. Finally, simple ordered semigroups are characterized through tripolar fuzzy simple.

We now establish numerous conceptions of tripolar fuzzy sets in ordered semigroups since the origin of tripolar fuzzy sets was first defined in Γ-semigroups.

Definition 3.1. *Let S be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a* tripolar fuzzy subsemigroup *of* **S** *if it satisfied the following conditions: for any $x, y \in S$,*

(1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$;
(2) $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$;
(3) $\delta_A(xy) \leq \max\{\delta_A(x), \delta_A(y)\}$.

The following are some more strongly concepts for tripolar fuzzy subsemigroups.

Definition 3.2. *Let* **S** *be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a* tripolar fuzzy left (resp. right) ideal *of* **S** *if it satisfied the following conditions: for every $x, y \in S$,*

(1) $\mu_A(xy) \geq \mu_A(y)$ *(resp.* $\mu_A(xy) \geq \mu_A(x)$*);*
(2) $\lambda_A(xy) \leq \lambda_A(y)$ *(resp.* $\lambda_A(xy) \leq \mu_A(x)$*);*
(3) $\delta_A(xy) \leq \delta_A(y)$ *(resp.* $\delta_A(xy) \leq \delta_A(x)$*);*
(4) $x \leq y$ *implies* $\mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$ *and* $\delta_A(x) \leq \delta_A(y)$.

A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a *tripolar fuzzy two-side ideal* (*or, tripolar fuzzy ideal*) if it is both a tripolar fuzzy left and a tripolar fuzzy right ideal of **S**.

We define a critical idea that is important in this work.

Definition 3.3. *Let* **S** *be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a* tripolar fuzzy interior ideal *of* **S** *if it satisfied the following conditions: for every $x, y, z \in S$,*

(1) $\mu_A(xyz) \geq \mu_A(y)$;
(2) $\lambda_A(xyz) \leq \lambda_A(y)$;
(3) $\delta_A(xyz) \leq \delta_A(y)$;
(4) $x \leq y$ *implies* $\mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$ *and* $\delta_A(x) \leq \delta_A(y)$.

The following proposition shows that the intersection of tripolar fuzzy interior ideals of **S** is also a tripolar fuzzy interior ideal of **S**.

Proposition 3.1. *Let* **S** *be an ordered semigroup. If $\{A_i \mid i \in I\}$ is a family of tripolar fuzzy interior ideals of* **S***, then a tripolar fuzzy subset $\bigcap_{i\in I} A_i := \left(\bigcap_{i\in I} \mu_{A_i}, \bigcup_{i\in I} \lambda_{A_i}, \bigcup_{i\in I} \delta_{A_i}\right)$ of* **S** *is a tripolar fuzzy interior ideal of* **S** *and is defined by*

$$\left(\bigcap_{i\in I} \mu_{A_i}\right)(x) := \bigcap_{i\in I} \mu_{A_i}(x) := \inf\{\mu_{A_i}(x) \mid i \in I\},$$

$$\left(\bigcup_{i\in I} \lambda_{A_i}\right)(x) := \bigcup_{i\in I} \lambda_{A_i}(x) := \sup\{\lambda_{A_i}(x) \mid i \in I\},$$

and

$$\left(\bigcup_{i\in I}\delta_{A_i}\right)(x) := \bigcup_{i\in I}\delta_{A_i}(x) := \sup\{\delta_{A_i}(x) \mid i \in I\},$$

for all $x \in S$.

Proof: Let $x, y, z \in S$. We obtain that

$$\begin{aligned}\left(\bigcap_{i\in I}\mu_{A_i}\right)(xyz) &= \bigcap_{i\in I}\mu_{A_i}(xyz)\\ &= \inf\{\mu_{A_i}(xyz) \mid i \in I\}\\ &\geq \inf\{\mu_{A_i}(y) \mid i \in I\}\\ &= \bigcap_{i\in I}\mu_{A_i}(y)\\ &= \left(\bigcap_{i\in I}\mu_{A_i}\right)(y),\end{aligned}$$

$$\begin{aligned}\left(\bigcup_{i\in I}\lambda_{A_i}\right)(xyz) &= \bigcup_{i\in I}\lambda_{A_i}(xyz)\\ &= \sup\{\lambda_{A_i}(xyz) \mid i \in I\}\\ &\leq \sup\{\lambda_{A_i}(y) \mid i \in I\}\\ &= \bigcup_{i\in I}\lambda_{A_i}(y)\\ &= \left(\bigcup_{i\in I}\lambda_{A_i}\right)(y),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in I}\delta_{A_i}\right)(xyz) &= \bigcup_{i\in I}\delta_{A_i}(xyz)\\ &= \sup\{\delta_{A_i}(xyz) \mid i \in I\}\\ &\leq \sup\{\delta_{A_i}(y) \mid i \in I\}\\ &= \bigcup_{i\in I}\delta_{A_i}(y)\\ &= \left(\bigcup_{i\in I}\delta_{A_i}\right)(y).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. We obtain

$$\begin{aligned}\left(\bigcap_{i\in I}\mu_{A_i}\right)(x) &= \bigcap_{i\in I}\mu_{A_i}(x)\\ &= \inf\{\mu_{A_i}(x) \mid i \in I\}\\ &\geq \inf\{\mu_{A_i}(y) \mid i \in I\}\\ &= \bigcap_{i\in I}\mu_{A_i}(y)\end{aligned}$$

$$= \left(\bigcap_{i\in I}\mu_{A_i}\right)(y),$$

$$\begin{aligned}\left(\bigcup_{i\in I}\lambda_{A_i}\right)(x) &= \bigcup_{i\in I}\lambda_{A_i}(x)\\ &= \sup\{\lambda_{A_i}(x) \mid i\in I\}\\ &\leq \sup\{\lambda_{A_i}(y) \mid i\in I\}\\ &= \bigcup_{i\in I}\lambda_{A_i}(y)\\ &= \left(\bigcup_{i\in I}\lambda_{A_i}\right)(y),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in I}\delta_{A_i}\right)(x) &= \bigcup_{i\in I}\delta_{A_i}(x)\\ &= \sup\{\delta_{A_i}(x) \mid i\in I\}\\ &\leq \sup\{\delta_{A_i}(y) \mid i\in I\}\\ &= \bigcup_{i\in I}\delta_{A_i}(y)\\ &= \left(\bigcup_{i\in I}\delta_{A_i}\right)(y).\end{aligned}$$

Therefore, $\bigcap_{i\in I} A_i$ is a tripolar fuzzy interior ideal of $\mathbf{S}$. □

The following result illustrates a relationship between tripolar fuzzy ideals and tripolar fuzzy interior ideals in ordered semigroups.

Proposition 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup. Then every tripolar fuzzy ideal of* $\mathbf{S}$ *is a tripolar fuzzy interior ideal of* $\mathbf{S}$*.*

Proof: Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy ideal of $\mathbf{S}$ and $x, y, z \in S$. Then we obtain that

$$\mu_A(xyz) = \mu(x(yz)) \geq \mu_A(yz) \geq \mu_A(y),$$
$$\lambda_A(xyz) = \lambda(x(yz)) \leq \lambda_A(yz) \leq \lambda_A(y),$$

and

$$\delta_A(xyz) = \delta(x(yz)) \leq \delta_A(yz) \leq \delta_A(y).$$

Hence A is a tripolar fuzzy interior ideal of $\mathbf{S}$. □

We note that the converse of Proposition 3.2 may not be true in general. Corollary 3.1, Corollary 3.2, and Corollary 3.3 show that the concepts of tripolar fuzzy ideals and tripolar fuzzy interior ideals are the same thing in certain classes of ordered semigroups: regular, intra-regular, and semisimple.

An ordered semigroup $\mathbf{S}$ is *regular* if for each element $a \in S$ there exists element $x \in S$ such that $a \leq axa$. Firstly, we illustrate the converse of Proposition 3.2 for regular ordered semigroups.

Theorem 3.1. *Let* $\mathbf{S}$ *be a regular ordered semigroup. Then every tripolar fuzzy interior ideal of* $\mathbf{S}$ *is a tripolar fuzzy ideal of* $\mathbf{S}$*.*

Proof: Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy interior ideal of $\mathbf{S}$ and $a, b \in S$. Since $\mathbf{S}$ is regular, there exists $x \in S$ such that $a \leq axa$ and we obtain

$$\mu_A(ab) \geq \mu_A(axab) = \mu_A((ax)ab) \geq \mu_A(a),$$
$$\lambda_A(ab) \leq \lambda_A(axab) = \lambda_A((ax)ab) \leq \lambda_A(a),$$

and

$$\delta_A(ab) \leq \delta_A(axab) = \delta_A((ax)ab) \leq \delta_A(a).$$

Thus, A is a tripolar fuzzy right ideal of $\mathbf{S}$. Similarly, we can show that A is a tripolar fuzzy left ideal of $\mathbf{S}$. Therefore, A is a tripolar fuzzy ideal of $\mathbf{S}$. □

Combining Proposition 3.2 and Theorem 3.1, we have the following corollary.

Corollary 3.1. *In regular ordered semigroup the concepts of tripolar fuzzy ideals and tripolar fuzzy interior ideals coincide.*

An ordered semigroup $\mathbf{S}$ is called *intra-regular* if for each element $a \in S$ there exist elements $x, y \in S$ such that $a \leq xa^2y$. For intra-regular ordered semigroups, the other direction of Proposition 3.2 is provided below.

Theorem 3.2. *Let* $\mathbf{S}$ *be an intra-regular ordered semigroup. Then every a tripolar fuzzy interior ideal of* $\mathbf{S}$ *is a tripolar fuzzy ideal of* $\mathbf{S}$.

Proof: Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy interior ideal of $\mathbf{S}$ and $a, b \in S$. Since $\mathbf{S}$ is an intra-regular ordered semigroup, there exist $x, y \in S$ such that $a \leq xa^2y$. Then $ab \leq xa^2yb = (xa)a(yb)$. Since A is tripolar fuzzy interior ideal, we have that

$$\mu_A(ab) \geq \mu_A\left(xa^2yb\right) = \mu_A((xa)a(yb)) \geq \mu_A(a),$$
$$\lambda_A(ab) \leq \lambda_A\left(xa^2yb\right) = \lambda_A((xa)a(yb)) \leq \lambda_A(a),$$

and

$$\delta_A(ab) \leq \delta_A\left(xa^2yb\right) = \delta_A((xa)a(yb)) \leq \delta_A(a).$$

Thus, A is a tripolar fuzzy right ideal of $\mathbf{S}$. Similarly, we can prove A is also a tripolar fuzzy left ideal of $\mathbf{S}$. Therefore, A is a tripolar fuzzy ideal of $\mathbf{S}$. □

We obtain the coincidence of tripolar fuzzy ideals and tripolar fuzzy interior ideals by combining Proposition 3.2 and Theorem 3.2.

Corollary 3.2. *In intra-regular ordered semigroup, the concepts of tripolar fuzzy ideals and tripolar fuzzy interior ideals coincide.*

An ordered semigroup $\mathbf{S}$ is called *semisimple* if for each element $a \in S$ there exist elements $x, y, z \in S$ such that $a \leq xayaz$. If an ordered semigroup is semisimple, the following theorem demonstrates the converse of Proposition 3.2.

Theorem 3.3. *Let* $\mathbf{S}$ *be a semisimple ordered semigroup. Then every tripolar fuzzy interior ideal of* $\mathbf{S}$ *is a tripolar fuzzy ideal of* $\mathbf{S}$.

Proof: Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy interior ideal of $\mathbf{S}$ and $a, b \in S$. Since $\mathbf{S}$ is a semisimple ordered semigroup, there exist $x, y, z \in S$ such that $a \leq xayaz$. Then $ab \leq xayazb = (xay)a(zb)$. Since A is a tripolar fuzzy interior ideal of $\mathbf{S}$, we have that

$$\mu_A(ab) \geq \mu_A(xayazb) = \mu_A((xay)a(zb)) \geq \mu_A(a),$$
$$\lambda_A(ab) \leq \lambda_A(xayazb) = \lambda_A((xay)a(zb)) \leq \lambda_A(a),$$

and

$$\delta_A(ab) \leq \delta_A(xayazb) = \delta_A((xay)a(zb)) \leq \delta_A(a).$$

Thus, A is a tripolar fuzzy right ideal of $\mathbf{S}$. Similarly, we can prove that A is a tripolar fuzzy left ideal of $\mathbf{S}$. Therefore, A is a tripolar fuzzy ideal of $\mathbf{S}$. □

Proposition 3.2 and Theorem 3.3 lead in the following corollary.

Corollary 3.3. *In semisimple ordered semigroup, the concepts of tripolar fuzzy ideals and tripolar fuzzy interior ideals coincide.*

In terms of tripolar fuzzy ideals, the following remark helps characterize left (right, and interior) ideals in ordered semigroups.

Remark 3.1. *Let X be a subset of S and $\chi_X = (\mu_{\chi_X}, \lambda_{\chi_X}, \delta_{\chi_X})$ a tripolar fuzzy subset of X in S. Suppose that one of the following statements holds:*
(1) $(\mu_{\chi_X})(x) = 1$;
(2) $(\lambda_{\chi_X})(x) = 0$;
(3) $(\delta_{\chi_X})(x) = -1$.
Then $x \in X$.

The following lemma illustrates a characterization of left ideals in ordered semigroups using tripolar fuzzy left ideals.

Lemma 3.1. *Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent:*
(1) L is a left ideal of $\mathbf{S}$;
(2) $\chi_L = (\mu_{\chi_L}, \lambda_{\chi_L}, \delta_{\chi_L})$ is a tripolar fuzzy left ideal of $\mathbf{S}$.

Proof: (1)⇒(2). Let $x, y \in S$. Since L is a left ideal of $\mathbf{S}$, we have $\mu_{\chi_L}(xy) = 1 = \mu_{\chi_L}(y)$, $\lambda_{\chi_L}(xy) = 0 = \lambda_{\chi_L}(y)$ and $\delta_{\chi_L}(xy) = -1 = \delta_{\chi_L}(y)$ whenever $y \in L$. If $y \notin L$, we obtain $\mu_{\chi_L}(xy) \geq 0 = \mu_{\chi_L}(y)$, $\lambda_{\chi_L}(xy) \leq 1 = \lambda_{\chi_L}(y)$ and $\delta_{\chi_L}(xy) \leq 0 = \delta_{\chi_L}(y)$.

Let $x, y \in S$ be such that $x \leq y$. Since L is a left ideal of $\mathbf{S}$, we have $x \in L$ and then $\mu_{\chi_L}(x) = 1 = \mu_{\chi_L}(y)$ whenever $y \in L$. If $y \notin L$, we have $\mu_{\chi_L}(x) \geq 0 = \mu_{\chi_L}(y)$, $\lambda_{\chi_L}(x) \leq 1 = \lambda_{\chi_L}(y)$ and $\delta_{\chi_L}(x) \leq 0 = \delta_{\chi_L}(y)$. Therefore, we obtain $\chi_L = (\mu_{\chi_L}, \lambda_{\chi_L}, \delta_{\chi_L})$ is a tripolar fuzzy left ideal of $\mathbf{S}$.

(2)⇒(1). Let $x \in S$ and $y \in L$. Since $\chi_L = (\mu_{\chi_L}, \lambda_{\chi_L}, \delta_{\chi_L})$ is a tripolar fuzzy left ideal of $\mathbf{S}$, we have $1 \geq \mu_{\chi_L}(xy) \geq \mu_{\chi_L}(y) = 1$. This implies that $\mu_{\chi_L}(xy) = 1$. By Remark 3.1, we have $xy \in L$. Therefore, $SL \subseteq L$.

Let $x, y \in S$ be such that $x \leq y$. If $y \in L$, since $\chi_L = (\mu_{\chi_L}, \lambda_{\chi_L}, \delta_{\chi_L})$ is a tripolar fuzzy left ideal of $\mathbf{S}$, we have $1 \geq \mu_{\chi_L}(x) \geq \mu_{\chi_L}(y) = 1$. This implies that $\mu_{\chi_L}(x) = 1$ and by Remark 3.1, we obtain $x \in L$. Therefore, L is a left ideal of $\mathbf{S}$. □

We can prove the following lemma using similar arguments used in Lemma 3.1.

Lemma 3.2. *Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent:*
(1) R is a right ideal of $\mathbf{S}$;
(2) $\chi_R = (\mu_{\chi_R}, \lambda_{\chi_R}, \delta_{\chi_R})$ is a tripolar fuzzy right ideal of $\mathbf{S}$.

By Lemma 3.1 and Lemma 3.2, we obtain the following corollary.

Corollary 3.4. *Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent:*
(1) D is an ideal of $\mathbf{S}$;
(2) $\chi_D = (\mu_{\chi_D}, \lambda_{\chi_D}, \delta_{\chi_D})$ is a tripolar fuzzy ideal of $\mathbf{S}$.

We now characterize interior ideals by using tripolar fuzzy interior ideals as follows.

Lemma 3.3. *Let $\mathbf{S}$ be an ordered semigroup and I a nonempty subset of S. Then the following statements are equivalent:*
(1) I is an interior ideal of $\mathbf{S}$;
(2) $\chi_I = (\mu_{\chi_I}, \lambda_{\chi_I}, \delta_{\chi_I})$ is a tripolar fuzzy interior ideal of $\mathbf{S}$.

Proof: (1)⇒(2). Let I be an interior ideal of $\mathbf{S}$ and $x, y, a \in S$. Suppose that $a \in I$. Since I is interior ideal, we have that $(\mu_{\chi_I})(xay) = 1 = (\mu_{\chi_I})(a)$, $(\lambda_{\chi_I})(xay) = 0 = (\lambda_{\chi_I})(a)$ and $(\delta_{\chi_I})(xay) = -1 = (\delta_{\chi_I})(a)$. If $a \notin I$, we have that $(\mu_{\chi_I})(xay) \geq 0 = (\mu_{\chi_I})(a)$, $(\lambda_{\chi_I})(xay) \leq 1 = (\lambda_{\chi_I})(a)$ and $(\delta_{\chi_I})(xay) \leq 0 = (\delta_{\chi_I})(a)$.

Let $x, y \in S$ be such that $x \leq y$. If $y \notin I$, we obtain $(\mu_{\chi_I})(x) \geq 0 = (\mu_{\chi_I})(y)$, $(\lambda_{\chi_I})(x) \leq 1 = (\lambda_{\chi_I})(y)$ and $(\delta_{\chi_I})(x) \leq 0 = (\delta_{\chi_I})(y)$. Since I is interior ideal, we have that $x \in I$ whenever $y \in I$. Then $(\mu_{\chi_I})(y) = 1 = (\mu_{\chi_I})(x)$, $(\lambda_{\chi_I})(y) = 0 = (\lambda_{\chi_I})(x)$ and $(\delta_{\chi_I})(y) = -1 = (\delta_{\chi_I})(x)$. Thus, $\chi_I = (\mu_{\chi_I}, \lambda_{\chi_I}, \delta_{\chi_I})$ is a tripolar fuzzy interior ideal of $\mathbf{S}$.

(2)⇒(1). Let $x, y \in S$ and $a \in I$. By hypothesis, we obtain $(\mu_{\chi_I})(xay) \geq (\mu_{\chi_I})(a) = 1$. This implies that $(\mu_{\chi_I})(xay) = 1$. By Remark 3.1, we have that $xay \in I$, that is $SIS \subseteq I$. Let $x, y \in S$ be such that $x \leq y$. If $y \in I$, we obtain $(\mu_{\chi_I})(x) \geq (\mu_{\chi_I})(y) = 1$ and then $(\mu_{\chi_I})(x) = 1$. By Remark 3.1, we obtain $x \in I$. Therefore, I is an interior ideal of $\mathbf{S}$. □

The complexity of tripolar fuzzy ideals is demonstrated in Lemma 3.1, Lemma 3.2, Lemma 3.3, and Corollary 3.4. According to our results, any ideals in ordered semigroups can only be represented by tripolar fuzzy sets with two outputs.

A series of lemmas is presented as a helpful tool for describing semisimple ordered semigroups.

Lemma 3.4. *Let $\mathbf{S}$ be an ordered semigroup and X, Y are subsets of S. Then the following statements hold:*

(1) $\chi_X = \chi_Y$ if and only if $X = Y$;
(2) $\chi_X \circ \chi_Y = \chi_{(XY]}$;
(3) $\chi_X \cap \chi_Y = \chi_{A \cap B}$.

Proof: Let X, Y be subsets of S. We give only the proof of (2) since the rests are not difficult to verify. If $a \in (XY]$, then there exist $x \in X$ and $y \in Y$ such that $a \leq xy$. This means that $\mathbf{S}_a \neq \emptyset$. Thus, we obtain

$$\begin{aligned}
(\mu_{\chi_X} \circ \mu_{\chi_Y})(a) &\leq 1 \\
&= \left(\mu_{\chi_{(XY]}}\right)(a) \\
&= \min\{(\mu_{\chi_X})(x), (\mu_{\chi_Y})(y)\} \\
&\leq \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{(\mu_{\chi_X})(p), (\mu_{\chi_Y})(q)\}\} \\
&= (\mu_{\chi_X} \circ \mu_{\chi_Y})(a), \\
(\lambda_{\chi_X} \circ \lambda_{\chi_Y})(a) &\geq 0 \\
&= \left(\lambda_{\chi_{(XY]}}\right)(a) \\
&= \max\{(\lambda_{\chi_X})(x), (\lambda_{\chi_Y})(y)\} \\
&\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{(\lambda_{\chi_X})(p), (\lambda_{\chi_Y})(q)\}\} \\
&= (\lambda_{\chi_X} \circ \lambda_{\chi_Y})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\delta_{\chi_X} \circ \delta_{\chi_Y})(a) &\geq -1 \\
&= \left(\delta_{\chi_{(XY]}}\right)(a) \\
&= \max\{(\delta_{\chi_X})(x), (\delta_{\chi_Y})(y)\}
\end{aligned}$$

$$\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{(\delta_{\chi_X})(p), (\delta_{\chi_Y})(q)\}\}$$
$$= (\delta_{\chi_X} \circ \delta_{\chi_Y})(a).$$

This implies that $\left(\mu_{\chi_{(XY]}}\right)(a) = (\mu_{\chi_X} \circ \chi_Y \mu_{\chi_Y})(a)$, $\left(\lambda_{\chi_{(XY]}}\right)(a) = (\lambda_{\chi_X} \circ \lambda_{\chi_Y})(a)$, and $\left(\delta_{\chi_{(XY]}}\right)(a) = (\delta_{\chi_X} \circ \delta_{\chi_Y})(a)$. If $a \notin (XY]$, then there is no $x, y \in S$ such that $a \leq xy$. This implies that $\mathbf{S}_a = \emptyset$. Thus, we obtain

$$\left(\mu_{\chi_{(XY]}}\right)(a) = 0 = (\mu_{\chi_X} \circ \mu_{\chi_Y})(a),$$
$$\left(\lambda_{\chi_{(XY]}}\right)(a) = 1 = (\lambda_{\chi_X} \circ \lambda_{\chi_Y})(a),$$

and

$$\left(\delta_{\chi_{(XY]}}\right)(a) = 0 = (\delta_{\chi_X} \circ \delta_{\chi_Y})(a).$$

Therefore, we have that $\chi_X \circ \chi_Y = \chi_{(XY]}$. □

The following result shows that the product of any two tripolar fuzzy interior ideals is a subset of their intersection.

Lemma 3.5. *Let* $\mathbf{S}$ *be a semisimple ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$, $B = (\mu_B, \lambda_B, \delta_B)$ *are tripolar fuzzy interior ideals of* $\mathbf{S}$*. Then* $A \circ B \subseteq A \cap B$.

Proof: Let A, B be tripolar fuzzy interior ideals of $\mathbf{S}$. Since $\mathbf{S}$ is semisimple, by Theorem 3.3, A, B are tripolar fuzzy ideals of $\mathbf{S}$. Let $a \in S$. Since $\mathbf{S}$ is semisimple, there exist $x, y, z \in S$ such that $a \leq xayaz$. This implies that $\mathbf{S}_a \neq \emptyset$. Then we obtain

$$\begin{aligned}
(\mu_A \circ \mu_B)(a) &= \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\mu_A(p), \mu_B(q)\}\} \\
&\leq \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\mu_A(pq), \mu_B(pq)\}\} \\
&\leq \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\mu_A(a), \mu_B(a)\}\} \\
&= \min\{\mu_A(a), \mu_B(a)\} \\
&= (\mu_A \cap \mu_B)(a), \\
(\lambda_A \circ \lambda_B)(a) &= \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\lambda_A(p), \lambda_B(q)\}\} \\
&\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\lambda_A(pq), \lambda_B(pq)\}\} \\
&\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\lambda_A(a), \lambda_B(a)\}\} \\
&= \max\{\lambda_A(a), \lambda_B(a)\} \\
&= (\lambda_A \cup \lambda_B)(a),
\end{aligned}$$

and

$$\begin{aligned}
(\delta_A \circ \delta_B)(a) &= \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\delta_A(p), \delta_B(q)\}\} \\
&\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\delta_A(pq), \delta_B(pq)\}\}
\end{aligned}$$

$$\begin{aligned}&\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\delta_A(a), \delta_B(a)\}\}\\&= \max\{\delta_A(a), \delta_B(a)\}\\&= (\delta_A \cup \delta_B)(a).\end{aligned}$$

Therefore, $A \circ B \subseteq A \cap B$. □

Semisimple ordered semigroups can be characterized in terms of ideals as follows.

Lemma 3.6 ([3])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is semisimple;*

(2) $I = \left(I^2\right]$ *for every ideal* I *of* $\mathbf{S}$*.*

Remark 3.2. *Every ideal of* $\mathbf{S}$ *is an interior ideal of* $\mathbf{S}$*.*

Now, we characterize semisimple ordered semigroups by using tripolar fuzzy interior ideals.

Theorem 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is semisimple;*

(2) $A \circ B = A \cap B$ *for every tripolar fuzzy interior ideals* $A = (\mu_A, \lambda_A, \delta_A)$ *and* $B = (\mu_B, \lambda_B, \delta_B)$ *of* $\mathbf{S}$*.*

Proof: (1)⇒(2). Let A and B be tripolar fuzzy interior ideals of $\mathbf{S}$. Since $\mathbf{S}$ is a semisimple ordered semigroup, B is a tripolar fuzzy ideal of $\mathbf{S}$. Let $a \in S$. Then there exist $x, y, z \in S$ such that $a \leq xayaz = (xay)(az)$. This implies that $\mathbf{S}_a \neq \emptyset$. Then

$$\begin{aligned}(\mu_A \circ \mu_B)(a) &= \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\mu_A(p), \mu_B(q)\}\}\\&\geq \min\{\mu_A(xay), \mu_B(az)\}\\&\geq \min\{\mu_A(a), \mu_B(a)\}\\&= (\mu_A \cap \mu_B)(a),\\(\lambda_A \circ \lambda_B)(a) &= \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\lambda_A(p), \lambda_B(q)\}\}\\&\leq \max\{\lambda_A(xay), \lambda_B(az)\}\\&\leq \max\{\lambda_A(a), \lambda_B(a)\}\\&= (\lambda_A \cup \lambda_B)(a),\end{aligned}$$

and

$$\begin{aligned}(\delta_A \circ \delta_B)(a) &= \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\delta_A(p), \delta_B(q)\}\}\\&\leq \max\{\delta_A(xay), \delta_B(az)\}\\&\leq \max\{\delta_A(a), \delta_B(a)\}\\&= (\delta_A \cup \delta_B)(a).\end{aligned}$$

Therefore, $A \circ B \supseteq A \cap B$. On the other hand by Lemma 3.5, $A \circ B \subseteq A \cap B$. Altogether we obtain $A \circ B = A \cap B$.

(2)⇒(1). Let I be an ideal of $\mathbf{S}$. By Remark 3.2, I is an interior ideal of $\mathbf{S}$ and by Lemma 3.3, we obtain χ_I is a tripolar fuzzy interior ideal of $\mathbf{S}$. By our presumption, we obtain

$$\chi_{(I^2]} = \chi_I \circ \chi_I = \chi_I \cap \chi_I = \chi_I.$$

By Lemma 3.4, we have $(I^2] = I$ and by Lemma 3.6, **S** is semisimple. □

By the above theorem, we obtain the following corollary immediately.

Corollary 3.5. *Let* **S** *be an ordered semigroup. Then the following statements are equivalent:*

(1) **S** *is semisimple;*
(2) $A = A \circ A$ *for every tripolar fuzzy interior ideal* A *of* **S**.

In contrast to Lemma 3.6 and Theorem 3.4, instead of using the concept of ideals to describe semisimple ordered semigroups, we can use a more general concept: tripolar fuzzy ideals.

We define the following useful notion in characterizing simple ordered semigroups. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of an ordered semigroup **S** and $a \in S$, we denote by $A(a)$ the subset of S defined as follows:

$$A(a) := \{b \in S \mid \mu_A(b) \geq \mu_A(a), \lambda_A(b) \leq \lambda_A(a) \text{ and } \delta_A(b) \leq \delta_A(a)\}.$$

We can see that $A(a)$ is not empty set since $a \in A(a)$.

Lemma 3.7. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* **S**. *If* A *is a tripolar fuzzy left ideal of* **S**, *then* $A(a)$ *is a left ideal of* **S** *for every* $a \in S$.

Proof: Let $a \in S$ and $b \in A(a)$. Since A is a tripolar fuzzy ideal of **S**, we have $\mu_A(ab) \geq \mu_A(b) \geq \mu_A(a)$, $\lambda_A(ab) \leq \lambda_A(b) \leq \lambda_A(a)$ and $\delta_A(ab) \leq \delta_A(b) \leq \delta_A(a)$. Then $ab \in A(a)$. Therefore, $SA(a) \subseteq A(a)$. Let $x, y \in S$ be such that $x \leq y$ and $y \in A(a)$. Since $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy left ideal of S, we have $\mu_A(x) \geq \mu_A(y) \geq \mu_A(a)$, $\lambda_A(x) \leq \lambda_A(y) \leq \lambda_A(a)$ and $\delta_A(x) \leq \delta_A(y) \leq \delta_A(a)$. This implies that $x \in A(a)$. Therefore, $A(a)$ is a left ideal of **S**. □

Similarly to Lemma 3.7, we can prove the following lemma.

Lemma 3.8. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* **S**. *If* A *is a tripolar fuzzy right ideal of* **S**, *then* $A(a)$ *is a right ideal of* **S** *for every* $a \in S$.

By Lemma 3.7 and Lemma 3.8, we obtain the following lemma.

Lemma 3.9. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* **S**. *If* A *is a tripolar fuzzy ideal of* **S**, *then* $A(a)$ *is an ideal of* **S** *for every* $a \in S$.

We recall the concept of simple ordered semigroups. An ordered semigroup **S** is called *simple* [3] if S does not contain proper ideals, that is, for every ideal D of **S**, we have $D = S$.

Definition 3.4. *An ordered semigroup* **S** *is called tripolar fuzzy simple if every tripolar fuzzy ideal of* **S** *is a tripolar fuzzy constant function, that is, for every tripolar fuzzy ideal* $A = (\mu_A, \lambda_A, \delta_A)$ *of* **S**, *we have* $\mu_A(a) = \mu_A(b)$, $\lambda_A(a) = \lambda_A(b)$ *and* $\delta_A(a) = \delta_A(b)$ *for all* $a, b \in S$.

Now, we characterize simple ordered semigroups by using tripolar fuzzy simple.

Theorem 3.5. *Let* **S** *be an ordered semigroup. Then the following statements are equivalent:*

(1) **S** *is simple;*
(2) **S** *is tripolar fuzzy simple.*

Proof: (1)⇒(2). Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy ideal of $\mathbf{S}$ and $a, b \in S$. By Lemma 3.9, we have $A(a)$ is an ideal of $\mathbf{S}$. Since S is simple, $A(a) = S$ and then $b \in A(a)$. This means that $\mu_A(b) \geq \mu_A(a)$, $\lambda_A(b) \leq \lambda_A(a)$ and $\delta_A(b) \leq \delta_A(a)$. Similarly, we can prove that $\mu_A(a) \geq \mu_A(b)$, $\lambda_A(a) \leq \lambda_A(b)$ and $\delta_A(a) \leq \delta_A(b)$. This implies $\mu_A(a) = \mu_A(b)$, $\lambda_A(a) = \lambda_A(b)$ and $\delta_A(a) = \delta_A(b)$. Therefore, A is a tripolar fuzzy constant function, and we obtain that $\mathbf{S}$ is tripolar fuzzy simple.

(2)⇒(1). Assume that S contains proper ideals and let D be an ideal of $\mathbf{S}$. By Lemma 3.3, we have $\chi_D = (\mu_{\chi_D}, \lambda_{\chi_D}, \delta_{\chi_D})$ is a tripolar fuzzy ideal of $\mathbf{S}$. Let $x \in S$. Since $\mathbf{S}$ is tripolar fuzzy simple, we have $\mu_{\chi_D}(x) = \mu_{\chi_D}(b)$, $\lambda_{\chi_D}(x) = \lambda_{\chi_D}(b)$ and $\delta_{\chi_D}(x) = \delta_{\chi_D}(b)$ for all $b \in S$. Let $a \in D$. Then $\mu_{\chi_D}(x) = \mu_{\chi_D}(a) = 1$. By Remark 3.1, we have $x \in D$. This implies that $S \subset D$. This is a contradiction. Therefore, $D = S$, and then $\mathbf{S}$ is simple. □

Before we leave this section, we provide an example of the notion of tripolar fuzzy interior ideals in ordered semigroups playing a vital role in this paper. This example demonstrates the effectiveness of tripolar fuzzification settings to the theory.

Example 3.1. *Let $S = \{a, b, c, d\}$. Define a binary operation $\cdot$ on S by following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

We define an order relation $\leq$ as follows:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

Now, $\mathbf{S} := (S; \cdot, \leq)$ is an ordered semigroup. We define tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S as follows:

S	μ_A	λ_A	δ_A
a	0.7	0.2	-0.8
b	0.4	0.5	-0.5
c	0.6	0.3	-0.3
d	0.2	0.7	-0.2

By careful calculation, we have that A is a tripolar fuzzy interior ideal of $\mathbf{S}$. If we assign $A = (\mu_A, \lambda_A)$, then A is an intuitionistic fuzzy interior ideal of $\mathbf{S}$, and if $A = (\mu_A, \delta_A)$, then A is a bipolar fuzzy interior ideal of $\mathbf{S}$ (see [5, 6]).

4. **Conclusion.**

The notion of tripolar fuzzy ideals was first introduced for Γ-semigroups. In this paper, we focus on the concept of tripolar fuzzy ideals in ordered semigroups. We introduced and studied various tripolar fuzzy ideals. We proved that tripolar fuzzy interior ideals and tripolar fuzzy ideals coincide in regular, intra-regular, and semisimple ordered semigroups. Using tripolar fuzzy interior ideals, the characterization of semisimple ordered semigroups was provided. Furthermore, we characterized simple ordered semigroups via tripolar fuzzy simple. We will apply these notions and results for studying related notions to algebraic hyperstructures in future work.

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