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REGULARITIES IN TERMS OF HYBRID n-INTERIOR IDEALS AND HYBRID (m, n)-IDEALS OF ORDERED SEMIGROUPS

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Abstract. *The notion of hybrid ideals is a generalization of ideals, (anti-) fuzzy ideals, and (int-)soft ideals in ordered semigroups. The concepts of interior ideals and bi-ideals can be extended to hybrid n-interior ideals and (m, n)-ideals, respectively. In this paper, we characterize fourteen regularities of ordered semigroups in terms of hybrid n-interior ideals and hybrid (m, n)-ideals in ordered semigroups.*
Keywords: Hybrid n-interior ideal, Hybrid (m, n)-ideal, Regular, Intra-regular, Ordered semigroup

1. **Introduction.** An ordered semigroup is an algebraic structure consisting of a semigroup, partially ordered relation on the semigroup that is compatible with the semigroup operation. In [11], Sanborisoot and Changphas introduced the concept of (m, n)-ideals of ordered semigroups and studied some characterizations of (m, n)-regular ordered semigroups in terms of (m, n)-ideals. Tiprachot et al. [13] gave the notion of n-interior ideals which is a generalization of interior ideals in 2022. Then, they also characterized fourteen regularities of ordered semigroups in terms of n-interior ideals and (m, n)-ideals.

In 2017, Phochai and Changphas [8] classified all regularities of ordered semigroups using linear inequations. In [13], the notion of n-interior ideals was defined and fourteen regularities of ordered semigroups were characterized in terms of n-interior ideals and (m, n)-ideals.

The notion of fuzzy sets was introduced by Zadeh [15] in 1965. In 1971, Rosenfeld [10] applied fuzzification to groups called fuzzy subgroups and then several researchers considered fuzzification on many algebraic structures such as on semigroups, semirings, rings, near-rings, ordered semigroups, and ordered semihypergroups. Fuzzy sets can also be used to study several topics in applied mathematics (see [12]). Later on, Molodtsov [6] gave the notion of soft sets as a new mathematical tool to deal with uncertainties.

Several years later, Jun et al. [3] gave the notion of hybrid structures which is a parallel circuit of fuzzy and soft sets. In [3], the notion of hybrid structures was introduced in

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a set of parameters over an initial universe set and then it was applied to BCK/BCI-algebras and linear spaces. We note that the name of BCK/BCI-algebras originated from the combinators B, C, I, and K in combinatory logic.

Anis et al. [1] provided the notion of hybrid subsemigroups and hybrid left (resp. right) ideals of semigroups and studied various properties. Elavarasan and Jun [2] gave the concept of hybrid bi-ideals of semigroups. They investigated some properties and studied characterizations of left (resp. right) simple semigroup and the completely regular semigroup using hybrid ideals and hybrid bi-ideals. In 2019, the notion of hybrid interior ideals of semigroup was introduced by Porselvi and Elavarasan [9] and it was shown that hybrid interior ideals and hybrid ideals coincide on regular semigroups and intra-regular semigroups. In 2021, Mekwian and Lekkoksung [7] applied hybrid structures on semigroups to ordered semigroups. They gave the notion of hybrid left (resp. right) ideals of ordered semigroups and provided their related properties.

In 2022, Tiprachot et al. [14] initialized the notion of hybrid n-interior ideals and hybrid (m,n)-ideals in ordered semigroups. Hybrid n-interior ideals turned out to be a generalization of the concepts of hybrid left and right ideals introduced by Mekwian and Lekkoksung. Similarly, hybrid (m,n)-ideals generalize fuzzy bi-ideals and soft bi-ideals. Furthermore, they gave some properties of hybrid n-interior (resp. (m,n)-) ideals and provided a connection between n-interior (resp. (m,n)-) ideals and hybrid n-interior (resp. (m,n)-) ideals in ordered semigroups.

We employ the merged concept of these two notions to describe ordered semigroups in this paper, motivated by the attribute in dealing with the uncertainty of fuzzy sets and soft sets. We are attempting to extend the results in [13] such that our results can handle non-constant conditions. More precisely, we characterize fourteen regularities of ordered semigroups in terms of hybrid n-interior ideals and hybrid (m,n)-ideals of ordered semigroups.

2. **Preliminaries.** An *ordered semigroup* is an algebraic structure $(S,\cdot,\leq)$ such that $(S,\cdot)$ is a semigroup, and $(S,\leq)$ is a partially ordered set such that $\leq$ is compatible with $\cdot$. That is, for any $x,y\in S$ such that $x\leq y$, we have that $x\cdot z\leq y\cdot z$ and $z\cdot x\leq z\cdot y$ for all $z\in S$.

For simplicity, we write xy instead of $x\cdot y$. Moreover, we denote an ordered semigroup $(S,\cdot,\leq)$ by its underlying set S.

Let S be an ordered semigroup, A, B be subsets of S, and C be a nonempty subset of S. Define the sets AB and $(C]$ as follows: $AB:=\emptyset$ if A or B is empty, otherwise,

$$AB:=\{ab : a\in A \text{ and } b\in B\} \text{ and } (C]:=\{x\in S : x\leq c \text{ for some } c\in C\}.$$

The following mentioned lemma is well-known in ordered semigroups.

Lemma 2.1. [4] *Let A and B be nonempty subsets of an ordered semigroup S. Then the following statements hold:*

(1) $A\subseteq (A]$ *and* $((A]]=(A]$;

(2) $A\subseteq B$ *implies* $(A]\subseteq (B]$;

(3) $(A](B]\subseteq (AB]$;

(4) $(A\cup B]=(A]\cup (B]$.

A *subsemigroup* of an ordered semigroup S means a nonempty subset A of S such that $AA\subseteq A$.

Definition 2.1. [5] *A subsemigroup A of an ordered semigroup S is called an* (m,n)-ideal *of S where $m,n\in\mathbb{N}_0$ if*

(1) $A^mSA^n\subseteq A$;

(2) *if $x\in A$ and $y\in S$ such that $y\leq x$, then $y\in A$.*

Since $A^0S = SA^0 = S$, then $A^0SA^n = SA^n$ and $A^mSA^0 = A^mS$. We observe that $(0,1)$-ideal is a left ideal of S, $(1,0)$-ideal is a right ideal of S and $(1,1)$-ideal is a bi-ideal of S.

Definition 2.2. [13] *A subsemigroup A of an ordered semigroup S is called an n-*interior ideal *of S where $n \in \mathbb{N}$ if*
(1) *$SA^nS \subseteq A$;*
(2) *if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.*

By the above definition, we can see that 1-interior ideal of S is interior ideal. Let A be a nonempty subset of S, we denote by $I_{(m,n)}(A)$ the (m,n)-ideal of S generated by A. One can show that $I_{(m,n)}(A) := (A \cup \cdots \cup A^{m+n} \cup A^mSA^n]$. If $A = \{a\}$, then we write $I_{(m,n)}(a)$ instead of $I_{(m,n)}(\{a\})$, (see [11]). Let $x \in S$. Then we define the new set as follows.

$$S_x := \{(a,b) \in S \times S \mid x \leq ab\}.$$

Let I be the unit interval $[0,1]$ and $\mathcal{P}(U)$ be the power set of an initial universal set U.

Definition 2.3. *A hybrid structure in S over U is defined as a mapping $\tilde{f}_\lambda := \left(\tilde{f}, \lambda\right) : S \to \mathcal{P}(U) \times I$, $x \mapsto \left(\tilde{f}(x), \lambda(x)\right)$, where $\tilde{f} : S \to \mathcal{P}(U)$ and $\lambda : S \to I$ are mappings.*

We denote $H(S)$ the set of all hybrid structures in S over U and define an order $\ll$ on $H(S)$ by letting $\tilde{f}_\lambda, \tilde{g}_\gamma \in H(S)$,

$$\tilde{f}_\lambda \ll \tilde{g}_\gamma \text{ if and only if } \tilde{f} \sqsubseteq \tilde{g},\ \lambda \succeq \gamma$$

where $\tilde{f} \sqsubseteq \tilde{g}$ means that $\tilde{f}(x) \subseteq \tilde{g}(x)$ and $\lambda \succeq \gamma$ means that $\lambda(x) \geq \gamma(x)$ for all $x \in S$. Moreover, $\tilde{f}_\lambda = \tilde{g}_\gamma$ if $\tilde{f}_\lambda \ll \tilde{g}_\gamma$ and $\tilde{g}_\gamma \ll \tilde{f}_\lambda$.

Definition 2.4. *Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in S over U. Then the intersection of hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f} \Cap \tilde{g}_\gamma$ and is defined as a hybrid structure $\tilde{f}_\lambda \Cap \tilde{g}_\gamma \colon S \to \mathcal{P}(U) \times I$, $x \mapsto \left(\left(\tilde{f} \cap \tilde{g}\right)(x), (\lambda \vee \gamma)(x)\right)$, where $\left(\tilde{f} \cap \tilde{g}\right)(x) := \tilde{f}(x) \cap \tilde{g}(x)$ and $(\lambda \vee \gamma)(x) := \max\{\lambda(x), \gamma(x)\}$.*

We denote $\tilde{S}_{\mathbf{S}}$ the hybrid structure $\tilde{S}_{\mathbf{S}} := \left(\tilde{S}, \mathbf{S}\right) : S \to \mathcal{P}(U) \times I$ defined by

$$\tilde{S}(x) := U \text{ and } \mathbf{S}(x) := 0, \text{ for all } x \in S.$$

For any hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ in S over U. The product of hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is defined as a hybrid structure $\tilde{f}_\lambda \otimes \tilde{g}_\gamma := \left(\tilde{f} \odot \tilde{g}, \lambda \circ \gamma\right) : S \to \mathcal{P}(U) \times I$, where

$$\left(\tilde{f} \odot \tilde{g}\right)(x) := \begin{cases} \bigcup\limits_{(u,v) \in S_x} \left[\tilde{f}(u) \cap \tilde{g}(v)\right] & \text{if } S_x \neq \emptyset, \\ \emptyset & \text{if } S_x = \emptyset, \end{cases}$$

and

$$(\lambda \circ \gamma)(x) := \begin{cases} \bigwedge\limits_{(u,v) \in S_x} \{\max\{\lambda(u), \gamma(v)\}\} & \text{if } S_x \neq \emptyset, \\ 1 & \text{if } S_x = \emptyset, \end{cases}$$

for all $x \in S$.

Let A be a nonempty subset of S. The characteristic hybrid structure in S over U is denoted by $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) := \left(\chi_A\left(\tilde{S}\right), \chi_A\left(\mathbf{S}\right)\right)$, where

$$\chi_A\left(\tilde{S}\right)(x) := \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A, \end{cases} \text{ and } \chi_A(\mathbf{S})(x) := \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A, \end{cases}$$

for all $x \in S$. If $A = S$, then we set $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) := \tilde{S}_{\mathbf{S}}$.

For $n \in \mathbb{N}$, we note that $a_1^n := a_1 a_2 \cdots a_n$ and $\tilde{f}_\lambda^{\,n} := \underbrace{\tilde{f}_\lambda \otimes \tilde{f}_\lambda \otimes \cdots \otimes \tilde{f}_\lambda}_{n \text{ times}}$, and this means that $\tilde{f}^n := \underbrace{\tilde{f} \odot \tilde{f} \odot \cdots \odot \tilde{f}}_{n \text{ times}}$ and $\lambda^n := \underbrace{\lambda \circ \lambda \circ \cdots \circ \lambda}_{n \text{ times}}$.

Now, we recall the concept of hybrid n-interior ideals and hybrid (m,n)-ideals of ordered semigroups defined in [14].

Definition 2.5. *Let S be an ordered semigroup. A hybrid subsemigroup $\tilde{f}_\lambda$ in S over U is called a* hybrid n-interior ideal in S over U *if for any $x, y, z_j \in S$, where $j \in \{1, \ldots, n\}$, and we have*

(1) $\tilde{f}(xz_1^n y) \supseteq \tilde{f}(z_1) \cap \cdots \cap \tilde{f}(z_n)$;

(2) $\lambda(xz_1^n y) \leq \max\{\lambda(z_1), \ldots, \lambda(z_n)\}$;

(3) $x \leq y$ *implies* $\tilde{f}(x) \supseteq \tilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$.

By Definition 2.5, we note that

(1) a mapping $\tilde{f}\colon S \to \mathcal{P}(U)$ is called an *int-soft n-interior ideal in S over U* if it satisfies (1) and $\tilde{f}(x) \supseteq \tilde{f}(y)$ whenever $x \leq y$;

(2) a mapping $\lambda\colon S \to I$ is called an *anti-fuzzy n-interior ideal of S* if it satisfies (2) and $\lambda(x) \leq \lambda(y)$ whenever $x \leq y$.

Definition 2.6. *Let S be an ordered semigroup and $m, n \in \mathbb{N}$. A hybrid subsemigroup $\tilde{f}_\lambda$ in S over U is called a* hybrid (m,n)-ideal in S over U *if for any $x, y_i, z_j \in S$, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we have*

(1) $\tilde{f}\left(y_1^m x z_1^n\right) \supseteq \left(\bigcap_{i=1}^m \tilde{f}(y_i)\right) \cap \left(\bigcap_{j=1}^n \tilde{f}(z_j)\right)$;

(2) $\lambda\left(y_1^m x z_1^n\right) \leq \max\{\lambda(y_i), \lambda(z_j) : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$;

(3) $x \leq y$ *implies* $\tilde{f}(x) \supseteq \tilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$.

By Definition 2.6, we note that

(1) a mapping $\tilde{f}\colon S \to \mathcal{P}(U)$ is called an *int-soft (m,n)-ideal in S over U* if it satisfies (1) and $\tilde{f}(x) \supseteq \tilde{f}(y)$ whenever $x \leq y$;

(2) a mapping $\lambda\colon S \to I$ is called an *anti-fuzzy (m,n)-ideal of S* if it satisfies (2) and $\lambda(x) \leq \lambda(y)$ whenever $x \leq y$.

We recall connections among characteristic hybrid structures, the relation $\Cap$ and the product $\otimes$.

Lemma 2.2. [14] *Let S be an ordered semigroup and A, B nonempty subsets of S. Then the following statements hold:*

(1) $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_B\left(\tilde{S}_{\mathbf{S}}\right) = \chi_{A \cap B}\left(\tilde{S}_{\mathbf{S}}\right)$;

(2) $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_B\left(\tilde{S}_{\mathbf{S}}\right) = \chi_{(AB]}\left(\tilde{S}_{\mathbf{S}}\right)$.

By Lemma 2.2, we obtain a connection between the usual subset relation and the relation $\ll$ on the set of all hybrid structures in ordered semigroups.

Lemma 2.3. [14] *Let A, B be nonempty subsets of an ordered semigroup S. Then the following statements are equivalent:*

(1) *$A \subseteq B$;*

(2) $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_B\left(\tilde{S}_{\mathbf{S}}\right)$.

By the above auxiliary lemmas, the characterizations of n-interior ideals and (m,n)-interior ideals in ordered semigroups by characteristic hybrid structures were obtained.

Theorem 2.1. [14] *Let A be nonempty subset of an ordered semigroup S. Then the following statements are equivalent:*

(1) *A is an n-interior ideal of S;*

(2) *$\chi_A\left(\tilde{S}_{\mathbf{S}}\right)$ is a hybrid n-interior ideal in S over U.*

Theorem 2.2. [14] *Let A be nonempty subset of an ordered semigroup S. Then the following statements are equivalent:*

(1) *A is an (m,n)-ideal of S;*

(2) *$\chi_A\left(\tilde{S}_{\mathbf{S}}\right)$ is a hybrid (m,n)-ideal in S over U.*

3. **Characterizations of Some Regularities.**

In this section, we characterize fourteen classes of ordered semigroups classified by linear inequations in terms of hybrid n-interior ideals and hybrid (m,n)-ideals in S over U.

An ordered semigroup S is *intra-quasi-regular* if for each $a \in S$ there exist $x, y, z \in S$ such that $a \leq xayaz$. Intra-quasi-regular ordered semigroups were characterized using the notion of ideals as follows.

Lemma 3.1. [13] *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is intra-quasi-regular;*

(2) *$I \cap B \subseteq (IBI]$ for any 1-interior ideal I and $(1,1)$-ideal B of S.*

With the above lemma and with the help of Theorems 2.1 and 2.2, we obtain a characterization of intra-quasi-regular ordered semigroups by hybrid ideals.

Theorem 3.1. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is intra-quasi-regular;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda \otimes \tilde{g}_\gamma$ for every hybrid 1-interior ideal $\tilde{g}_\gamma$ and every hybrid $(1,1)$-ideal $\tilde{f}_\lambda$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be a hybrid 1-interior ideal and a hybrid $(1,1)$-ideal in S over U, respectively. Let $a \in S$. Since S is intra-quasi-regular, there exist $x, y, z \in S$ such that $a \leq xayaz \leq x(xaya)y(xaya)z = (x_1ay)(ay_1a)(yaz)$ for some $x_1, y_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain

$$\begin{aligned}\left(\tilde{g} \odot \tilde{f} \odot \tilde{g}\right)(a) &= \bigcup_{(u,v)\in S_a} \left[\left(\tilde{g} \odot \tilde{f}\right)(u) \cap \tilde{g}(v)\right] \\ &\supseteq \left(\tilde{g} \odot \tilde{f}\right)(x_1ayay_1a) \cap \tilde{g}(yaz)\end{aligned}$$

$$= \left[\bigcup_{(p,q)\in S_{x_1ayay_1a}} \tilde{g}(p) \cap \tilde{f}(q) \right] \cap \tilde{g}(yaz)$$
$$\supseteq \tilde{g}(x_1ay) \cap \tilde{f}(ay_1a) \cap \tilde{g}(yaz)$$
$$\supseteq \tilde{g}(a) \cap \left(\tilde{f}(a) \cap \tilde{f}(a)\right) \cap \tilde{g}(a)$$
$$= \left(\tilde{g} \cap \tilde{f}\right)(a).$$

This implies that $\tilde{g} \cap \tilde{f} \sqsubseteq \tilde{g} \odot \tilde{f} \odot \tilde{g}$. Consider

$$(\gamma \circ \lambda \circ \gamma)(a) = \bigwedge_{(u,v)\in S_a} \{\max\{(\gamma \circ \lambda)(u), \gamma(v)\}\}$$
$$\leq \max\{(\gamma \circ \lambda)(x_1ayay_1a), \gamma(yaz)\}$$
$$= \max\left\{ \max\left\{ \bigwedge_{(p,q)\in S_{x_1ayay_1a}} \tilde{g}(p), \tilde{f}(q) \right\}, \tilde{g}(yaz) \right\}$$
$$\leq \max\{\gamma(x_1ay), \lambda(ay_1a), \gamma(yaz)\}$$
$$\leq \max\{\gamma(a), \{\max\{\lambda(a), \lambda f(a)\}\}, \gamma(a)\}$$
$$= (\lambda \vee \gamma)(a).$$

This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda \circ \gamma$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda \otimes \tilde{g}_\gamma$.

(2) $\Rightarrow$ (1). Let I and B be a 1-interior ideal and a $(1,1)$-ideal of S, respectively. Then, by Theorem 2.1 and by Theorem 2.2, $\chi_I\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_B\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid 1-interior ideal and a hybrid $(1,1)$-ideal in S over U, respectively. By assumption, we have $\chi_I\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_B\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_I\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_B\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_I\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{I\cap B}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(IBI]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $I \cap B \subseteq (IBI]$. Then, by Lemma 3.1, S is an intra-quasi-regular ordered semigroup. □

An ordered semigroup S is *left-quasi-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xaya$. The notions of bi-ideals and interior ideals were used to characterize left-quasi-regular ordered semigroups as follows.

Lemma 3.2. [13] *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is left-quasi-regular;*

(2) *$I \cap B \subseteq (IB]$ for every 1-interior ideal I and every $(1,1)$-ideal B of S.*

By the above result, we obtain the following characterization.

Theorem 3.2. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is left-quasi-regular;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid 1-interior ideal $\tilde{g}_\gamma$ and every hybrid $(1,1)$-ideal $\tilde{f}_\lambda$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be a hybrid 1-interior ideal and a hybrid $(1,1)$-ideal in S over U, respectively. Let $a \in S$. Since S is left-quasi-regular, there exist $x, y \in S$ such that $a \leq xaya \leq x(xaya)y(xaya) = (x_1ay_1)(aya)$ for some $x_1, y_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain $\left(\tilde{g} \odot \tilde{f}\right)(a) = \bigcup_{(u,v)\in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right] \supseteq \tilde{g}(x_1 a y_1) \cap \tilde{f}(aya) \supseteq \tilde{g}(a) \cap \left(\tilde{f}(a) \cap \tilde{f}(a)\right) = \left(\tilde{g} \cap \tilde{f}\right)(a)$. This implies that $\tilde{g} \cap \tilde{f} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider $(\gamma \circ \lambda)(a) = \bigwedge_{(u,v)\in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \leq \max\{\gamma(x_1 a y_1), \lambda(aya)\} \leq \max\{\gamma(a), \max\{\lambda(a), \lambda(a)\}\} = (\gamma \vee \lambda)(a)$. This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$.

(2) $\Rightarrow$ (1). Let I and B be a 1-interior ideal and a $(1, 1)$-ideal of S, respectively. Then, by Theorem 2.1 and by Theorem 2.2, $\chi_I\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_B\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid 1-interior ideal and a hybrid $(1, 1)$-ideal in S over U, respectively. By assumption, we have $\chi_I\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_B\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_I\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_B\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{I\cap B}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(IB]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $I \cap B \subseteq (IB]$. Then, by Lemma 3.2, S is a left-quasi-regular ordered semigroup. □

An ordered semigroup S satisfies $C8$ if for each $a \in S$ there exist $x, y \in S$ such that $a \leq x a^k y$ where $k \in \mathbb{N}$. Ordered semigroups satisfying $C8$ were described by the following lemma.

Lemma 3.3. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C8$;*

(2) *$A \cap E \subseteq (AE]$ for every k-interior ideals A and E of S.*

Proof: This can be proved similarly as Theorem 3.10 in [13]. □

The following theorem uses Lemma 3.3 to represent ordered semigroups satisfying $C8$ by hybrid n-interior ideals.

Theorem 3.3. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C8$;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid k-interior ideals $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be hybrid k-interior ideals in S over U. Let $a \in S$. Since S satisfies $C8$, there exist $x, y \in S$ such that $a \leq x a^k y \leq x\left(x a^k y\right)\left(x a^k y\right) a^{k-2} y a = \left(x_1 a^k y\right)\left(x a^k y_1\right)$ for some $x_1, y_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain

$$\begin{aligned}\left(\tilde{g} \odot \tilde{f}\right)(a) &= \bigcup_{(u,v)\in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right] \\ &\supseteq \tilde{g}\left(x_1 a^k y\right) \cap \tilde{f}\left(x a^k y_1\right) \\ &\supseteq (\underbrace{\tilde{g}(a) \cap \cdots \cap \tilde{g}(a)}_{k \text{ times}}) \cap (\underbrace{\tilde{f}(a) \cap \cdots \cap \tilde{f}(a)}_{k \text{ times}}) \\ &= \left(\tilde{g} \cap \tilde{f}\right)(a).\end{aligned}$$

This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider

$$\begin{aligned}(\gamma \circ \lambda)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \\ &\leq \max\left\{\gamma\left(x_1 a^k y\right), \lambda\left(x a^k y_1\right)\right\} \\ &\leq \max\left\{\max\{\underbrace{\gamma(a), \ldots, \gamma(a)}_{k \text{ times}}\}, \max\{\underbrace{\lambda(a), \ldots, \lambda(a)}_{k \text{ times}}\}\right\}\end{aligned}$$

$$= (\gamma \vee \lambda)(a).$$

This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$.

(2) $\Rightarrow$ (1). Let A and E be a k-interior ideals of S. Then, by Theorem 2.1, $\chi_A\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_E\left(\tilde{S}_{\mathbf{S}}\right)$ are hybrid k-interior ideals in S over U. By assumption, we have $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_E\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_A\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_E\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{A\cap E}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(AE]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $A \cap E \subseteq (AE]$. Thus, S satisfies $C8$ by Lemma 3.3. □

An ordered semigroup S is said to be

(1) *intra-reproduce* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xay$;
(2) *intra-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$.

It is uncomplicated to see that intra-reproduce ordered semigroups meet $C8$ with $k = 1$ and intra-regular ordered semigroups satisfy $C8$ with $k = 2$. We have the following consequences by plugging $k = 1$ and $k = 2$ in Theorem 3.3.

Corollary 3.1. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S is intra-reproduce;*
(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid k-interior ideals $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ in S over U.*

Proof: It is clear by applying $k = 1$ in Theorem 3.3. □

Corollary 3.2. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S is intra-regular;*
(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid k-interior ideals $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ in S over U.*

Proof: It is clear by applying $k = 2$ in Theorem 3.3. □

An ordered semigroup S satisfies $C9$ if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^kya$, where $k \in \mathbb{N}$. Generalizations of interior ideals and bi-ideals were used to illustrate a representation of ordered semigroups satisfying $C9$.

Lemma 3.4. [13] *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C9$;*
(2) *$A \cap J \subseteq (AJ]$ for every k-interior ideal A and every $(k, 1)$-ideal J of S.*

We are now able to describe ordered semigroups satisfying $C9$ by hybrid ideals.

Theorem 3.4. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C9$;*
(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid k-interior ideal $\tilde{g}_\gamma$ and every hybrid $(k, 1)$-ideal $\tilde{f}_\lambda$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be a hybrid k-interior ideal and a hybrid $(k, 1)$-ideal in S over U, respectively. Let $a \in S$. Since S satisfies $C9$, there exist $x, y \in S$ such that $a \leq xa^kya \leq x\left(xa^kya\right)\left(xa^kya\right)a^{k-2}ya = \left(x_1a^kx_2\right)\left(a^ky_1a\right)$ for some $x_1, x_2, y_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain

$$\left(\tilde{g} \odot \tilde{f}\right)(a) = \bigcup_{(u,v)\in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right]$$
$$\supseteq \tilde{g}\left(x_1a^kx_2\right) \cap \tilde{f}\left(a^ky_1a\right)$$

$$\supseteq (\underbrace{\tilde{g}(a) \cap \cdots \cap \tilde{g}(a)}_{k \text{ times}}) \cap \Big(\underbrace{\tilde{f}(a) \cap \cdots \cap \tilde{f}(a)}_{k+1 \text{ times}}\Big)$$
$$= \left(\tilde{g} \cap \tilde{f}\right)(a).$$

This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider

$$\begin{aligned}(\gamma \circ \lambda)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \\ &\le \max\left\{\gamma\left(x_1 a^k x_2\right), \lambda\left(a^k y_1 a\right)\right\} \\ &\le \max\left\{\max\{\underbrace{\gamma(a), \ldots, \gamma(a)}_{k \text{ times}}\}, \max\{\underbrace{\lambda(a), \ldots, \lambda(a)}_{k+1 \text{ times}}\}\right\} \\ &= (\gamma \vee \lambda)(a).\end{aligned}$$

This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$.

(2) $\Rightarrow$ (1). Let A and J be a k-interior ideal and a $(k,1)$-ideal of S, respectively. By Theorem 2.1 and Theorem 2.2, $\chi_A\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_J\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid k-interior ideal and a hybrid $(k,1)$-ideal in S over U, respectively. By assumption, we have $\chi_A\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_J\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_A\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_J\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{A\cap J}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(AJ]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $A \cap J \subseteq (AJ]$. Therefore, S satisfies $C9$ by Lemma 3.4. □

An ordered semigroup S satisfies $C10$ if for each $a \in S$ there exist $x, y \in S$ such that $a \le axa^k y$ where $k \in \mathbb{N}$. The concepts of n-interior ideals and (m,n)-ideals were used to describe ordered semigroups satisfying $C10$ as follows.

Lemma 3.5. [13] *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C10$;*

(2) *$A \cap F \subseteq (FA]$ for every k-interior ideal A and every $(1,k)$-ideal F of S.*

Similar to Theorem 3.4, we have the following theorem.

Theorem 3.5. *Let S be an ordered semigroup and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *S satisfies $C10$;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid k-interior ideal $\tilde{g}_\gamma$ and every hybrid $(1,k)$-ideal $\tilde{f}_\lambda$ in S over U.*

An ordered semigroup S satisfies $C11$ if for each $a \in S$ there exist $x, y \in S$ such that $a \le axa^k ya$ where $k \in \mathbb{N}$. The following lemma illustrates a characterization of ordered semigroups satisfying $C11$ by (m,n)-ideals.

Lemma 3.6. *Let S be an ordered semigroup and $p, q \in \mathbb{N}$ such that $p + q = k$. Then the following statements are equivalent:*

(1) *S satisfies $C11$;*

(2) *$G \cap H \subseteq (GH]$ for every $(1,p)$-ideal G and every $(q,1)$-ideal H of S.*

Proof: (1) $\Rightarrow$ (2). Let G and H be a $(1,p)$-ideal and a $(q,1)$-ideal of S, respectively. Then $G\cap H \subseteq ((G\cap H)S(G\cap H)^p(G\cap H)^q S(G\cap H)] \subseteq (GSG^pH^qSH] \subseteq (GH]$. Therefore, $G \cap H \subseteq (GH]$.

(2) $\Rightarrow$ (1). Let $a \in S$. Then

$$\begin{aligned} a &\in I_{(1,p)}(a) \cap I_{(q,1)}(a) \\ &\subseteq \left(I_{(1,p)}(a) I_{(q,1)}(a)\right] && \text{(by assumption)} \\ &= \left(\left(a \cup a^2 \cup \cdots \cup a^{p+1} \cup aSa^p\right] \left(a \cup a^2 \cup \cdots \cup a^{q+1} \cup a^q Sa\right]\right] \\ &\subseteq \left(a^2 \cup a^3 \cup \cdots \cup a^k \cup a^{k+2} \cup aSa^k Sa\right]. \end{aligned}$$

This implies that $a \leq t$ for some $t \in \left(a^2 \cup a^3 \cup \cdots \cup a^k \cup a^{k+2} \cup aSa^k Sa\right]$. For any case of t, we have that $a \leq axa^k ya$ for some $x, y \in S$. Therefore, S satisfies $C11$. □

Ordered semigroups satisfying $C11$ can be described by hybrid (m, n)-ideals in the help of Lemma 3.6.

Theorem 3.6. *Let S be an ordered semigroup and $p, q \in \mathbb{N}$ such that $p + q = k$. Then the following statements are equivalent:*

(1) S satisfies $C11$;

(2) $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid $(1, p)$-ideal $\tilde{g}_\gamma$ and every hybrid $(q, 1)$-ideal $\tilde{f}_\lambda$ in S over U.

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be a hybrid $(1, p)$-ideal and a hybrid $(q, 1)$-ideal in S over U, respectively. Let $a \in S$. Then, since S satisfies $C11$, there exist $x, y \in S$ such that $a \leq axa^k ya = (axa^p)(a^q ya)$. This implies that $S_a \neq \emptyset$. We obtain

$$\begin{aligned} \left(\tilde{g} \odot \tilde{f}\right)(a) &= \bigcup_{(u,v) \in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right] \\ &\supseteq \tilde{g}\left(axa^p\right) \cap \tilde{f}\left(a^q ya\right) \\ &\supseteq (\underbrace{\tilde{g}(a) \cap \cdots \cap \tilde{g}(a)}_{p+1 \text{ times}}) \cap (\underbrace{\tilde{f}(a) \cap \cdots \cap \tilde{f}(a)}_{q+1 \text{ times}}) \\ &= \left(\tilde{g} \cap \tilde{f}\right)(a). \end{aligned}$$

This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider

$$\begin{aligned} (\gamma \circ \lambda)(a) &= \bigwedge_{(u,v) \in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \\ &\leq \max\left\{\gamma\left(axa^p\right), \lambda\left(a^q xxya\right)\right\} \\ &\leq \max\left\{ \max\{\underbrace{\gamma(a), \ldots, \gamma(a)}_{p+1 \text{ times}}\}, \max\{\underbrace{\lambda(a), \ldots, \lambda(a)}_{q+1 \text{ times}}\} \right\} \\ &= (\gamma \vee \lambda)(a). \end{aligned}$$

This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$.

(2) $\Rightarrow$ (1). Let G and H be a $(1, p)$-ideal and a $(1, q)$-ideal of S, respectively. Then, by Theorem 2.2, $\chi_G\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_H\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid $(1, p)$-ideal and a hybrid $(q, 1)$-ideal in S over U, respectively. By assumption and Lemma 2.2, we obtain $\chi_{G \cap H}\left(\tilde{S}_{\mathbf{S}}\right) = \chi_G\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_H\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_G\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_H\left(\tilde{S}_{\mathbf{S}}\right) = \chi_{(GH]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $G \cap H \subseteq (GH]$. Then, by Lemma 3.6, S satisfies $C11$. □

An ordered semigroup S is *regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$. Interior ideals and bi-ideals characterized regular ordered semigroups by the following formation.

Lemma 3.7. [13] *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular;*
(2) *$I \cap B \subseteq (BIB]$ for any 1-interior ideal I and $(1,1)$-ideal B of S.*

The following result extends the previous lemma by allowing regular ordered semigroups to be represented using generalizations of interior ideals and bi-ideals.

Theorem 3.7. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is regular;*
(2) *$\tilde{g_\gamma} \Cap \tilde{f}_\lambda \ll \tilde{g_\gamma} \otimes \tilde{f}_\lambda \otimes \tilde{g_\gamma}$ for every hybrid 1-interior ideal $\tilde{f}_\lambda$ and every hybrid $(1,1)$-ideal $\tilde{g_\gamma}$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{f}_\lambda$ and $\tilde{g_\gamma}$ be a hybrid 1-interior ideal and a hybrid $(1,1)$-ideal in S over U, respectively. Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \le axa \le (axa)x(axa) = ax(axa)x(axa)xa = (axa)(x_1ax)(axa)$ for some $x_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain

$$\begin{aligned}
\left(\tilde{g} \odot \tilde{f} \odot \tilde{g}\right)(a) &= \bigcup_{(u,v)\in S_a} \left[\left(\tilde{g} \odot \tilde{f}\right)(u) \cap \tilde{g}(v)\right] \\
&\supseteq \left(\tilde{g} \odot \tilde{f}\right)(axax_1ax) \cap \tilde{g}(axa) \\
&= \left[\bigcup_{(p,q)\in S_{axax_1ax}} \tilde{g}(p) \cap \tilde{f}(q)\right] \cap \tilde{g}(axa) \\
&\supseteq \tilde{g}(axa) \cap \tilde{f}(x_1ax) \cap \tilde{g}(axa) \\
&\supseteq (\tilde{g}(a) \cap \tilde{g}(a)) \cap \tilde{f}(a) \cap (\tilde{g}(a) \cap \tilde{g}(a)) \\
&= \left(\tilde{f} \cap \tilde{g}\right)(a).
\end{aligned}$$

This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f} \odot \tilde{g}$. Consider

$$\begin{aligned}
(\gamma \circ \lambda \circ \gamma)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{(\gamma \circ \lambda)(u), \gamma(v)\}\} \\
&\le \max\{(\gamma \circ \lambda)(axax_1ax), \gamma(axa)\} \\
&= \max\left\{\max\left\{\bigwedge_{(p,q)\in S_{axax_1ax}} \tilde{g}(p), \tilde{f}(q)\right\}, \tilde{g}(axa)\right\} \\
&\le \max\{\gamma(axa), \lambda(xax), \gamma(axa)\} \\
&\le \max\{\{\max\{\gamma(a), \gamma(a)\}\}, \lambda(a), \{\max\{\gamma(a), \gamma(a)\}\}\} \\
&= (\lambda \vee \gamma)(a).
\end{aligned}$$

This implies that $\lambda \vee \gamma \succeq \gamma \circ \lambda \circ \gamma$. Therefore, $\tilde{g_\gamma} \Cap \tilde{f}_\lambda \ll \tilde{g_\gamma} \otimes \tilde{f}_\lambda \otimes \tilde{g_\gamma}$.

(2) $\Rightarrow$ (1). Let I and B be an 1-interior ideal and a $(1,1)$-ideal of S, respectively. Then, by Theorem 2.1 and by Theorem 2.2, $\chi_I\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_B\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid 1-interior ideal and a hybrid $(1,1)$-ideal in S over U, respectively. By assumption, we have $\chi_I\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_B\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_B\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_I\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_B\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{I\cap B}\left(\tilde{S}_{\mathbf{S}}\right) \ll$

$\chi_{(BIB]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $I \cap B \subseteq (BIB]$. Therefore, S is a regular ordered semigroup by Lemma 3.7. □

An ordered semigroup S is *left regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq xa^2$. This regularity can be characterized by (m, n)-ideals as follows.

Lemma 3.8. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is left regular;*

(2) *$L \cap C \subseteq (LC]$ for every $(0, 1)$-ideal L and every $(1, 2)$-ideal C of S.*

Proof: (1) ⇒ (2). Let L and C be a $(0, 1)$-ideal and a $(1, 2)$-ideal of S, respectively. Then $L \cap C \subseteq (S(L \cap C)(L \cap C)] \subseteq (SLC] \subseteq (LC]$. Therefore, $L \cap C \subseteq (LC]$.

(2) ⇒ (1). Let $a \in S$. Then

$$\begin{aligned} a &\in I_{(0,1)}(a) \cap I_{(1,2)}(a) \\ &\subseteq (I_{(0,1)}(a) I_{(1,2)}(a)] && \text{(by assumption)} \\ &= \left((a \cup Sa] \left(a \cup a^2 \cup a^3 \cup aSa^2\right]\right] \\ &\subseteq \left(a^2 \cup Sa^2\right]. \end{aligned}$$

This implies that $a \leq t$ for some $t \in \left(a^2 \cup Sa^2\right)$. For any case of t, we have that $a \leq xa^2$ for some $x \in S$. Therefore, S is left regular. □

We get the following theorem by applying Lemma 3.8.

Theorem 3.8. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is left regular;*

(2) *$\tilde{g_\gamma} \Cap \tilde{f_\lambda} \ll \tilde{g_\gamma} \otimes \tilde{f_\lambda}$ for every hybrid $(0, 1)$-ideal $\tilde{g_\gamma}$ and every hybrid $(1, 2)$-ideal $\tilde{f_\lambda}$ in S over U.*

Proof: (1) ⇒ (2). Let $\tilde{g_\gamma}$ and $\tilde{f_\lambda}$ be a hybrid $(0, 1)$-ideal and a hybrid $(1, 2)$-ideal in S over U, respectively. Let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq xa^2 \leq x\left(xa^2\right)\left(xa^2\right) = (x_1 a)\left(axa^2\right)$ for some $x_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain $\left(\tilde{g} \odot \tilde{f}\right)(a) = \bigcup_{(u,v) \in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right] \supseteq \tilde{g}(x_1 a) \cap \tilde{f}\left(axa^2\right) \supseteq \tilde{g}(a) \cap \left(\tilde{f}(a) \cap \tilde{f}(a) \cap \tilde{f}(a)\right) = \left(\tilde{g} \cap \tilde{f}\right)(a)$. This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider $(\gamma \circ \lambda)(a) = \bigwedge_{(u,v) \in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \leq \max\left\{\gamma(x_1 a), \lambda\left(axa^2\right)\right\} \leq \max\{\gamma(a), \max\{\lambda(a), \lambda(a), \lambda(a)\}\} = \max\{\gamma(a), \lambda(a)\} = (\gamma \vee \lambda)(a)$. This implies that $\gamma \vee \lambda \succeq \lambda \circ \gamma$. Therefore, $\tilde{g_\gamma} \Cap \tilde{f_\lambda} \ll \tilde{g_\gamma} \otimes \tilde{f_\lambda}$.

(2) ⇒ (1). Let L and C be a $(0, 1)$-ideal and a $(1, 2)$-ideal of S. Then, by Theorem 2.2, $\chi_L\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_C\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid $(0, 1)$-ideal and a hybrid $(1, 2)$-ideal in S over U, respectively. By assumption, we have $\chi_L\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_C\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_L\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_C\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{L \cap C}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(LC]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $L \cap C \subseteq (LC]$. Hence, by Lemma 3.8, S satisfies $C9$. □

An ordered semigroup S is *right regular* if for each $a \in S$ there exists $x \in S$ such that $a \leq a^2 x$. Similar to Theorem 3.8, we have the following lemma.

Lemma 3.9. [13] *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is right regular;*

(2) *$L \cap D \subseteq (DL]$ for every $(0, 1)$-ideal L and every $(2, 1)$-ideal D of S.*

Similar to Theorem 3.8, we have the following theorem.

Theorem 3.9. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is right regular;*
(2) $\tilde{g_\gamma} \Cap \tilde{f_\lambda} \ll \tilde{g_\gamma} \otimes \tilde{f_\lambda}$ *for every hybrid* $(2,1)$*-ideal* $\tilde{g_\gamma}$ *and every hybrid* $(0,1)$*-ideal* $\tilde{f_\lambda}$ *in* S *over* U*.*

An ordered semigroup S is *completely regular* if for each $a \in S$ there exists $x \in S$ such that $a \le a^2xa^2$. The following lemma uses the concept of (m,n)-ideals to characterize completely regular ordered semigroups.

Lemma 3.10. [13] *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is completely regular;*
(2) $D \cap C \subseteq (DC]$ *for every* $(2,1)$*-ideal* D *and every* $(1,2)$*-ideal* C *of* S*.*

By this lemma, we have the following result.

Theorem 3.10. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S is completely regular;*
(2) $\tilde{g_\gamma} \Cap \tilde{f_\lambda} \ll \tilde{g_\gamma} \otimes \tilde{f_\lambda}$ *for every hybrid* $(2,1)$*-ideal* $\tilde{g_\gamma}$ *and every hybrid* $(1,2)$*-ideal* $\tilde{f_\lambda}$ *in* S *over* U*.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g_\gamma}$ and $\tilde{f_\lambda}$ be a hybrid $(2,1)$-ideal and a hybrid $(1,2)$-ideal in S over U, respectively. Let $a \in S$. Since S is completely regular, there exists $x \in S$ such that $a \le a^2xa^2 \le a\left(a^2xa^2\right)x\left(a^2xa^2\right)a = \left(a^2x_1a\right)\left(ax_2a^2\right)$ for some $x_1, x_2 \in S$. This implies that $S_a \neq \emptyset$. We obtain $\left(\tilde{g} \odot \tilde{f}\right)(a) = \bigcup_{(u,v)\in S_a}\left[\tilde{g}(u) \cap \tilde{f}(v)\right] \supseteq \tilde{g}\left(a^2x_1a\right) \cap \tilde{f}\left(ax_2a^2\right) \supseteq (\tilde{g}(a) \cap \tilde{g}(a) \cap \tilde{g}(a)) \cap \left(\tilde{f}(a) \cap \tilde{f}(a) \cap \tilde{f}(a)\right) = \left(\tilde{g} \cap \tilde{f}\right)(a)$. This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider $(\gamma \circ \lambda)(a) = \bigwedge_{(u,v)\in S_a}\{\max\{\gamma(u), \lambda(v)\}\} \le \max\left\{\gamma\left(a^2x_1a\right), \lambda\left(ax_2a^2\right)\right\} \le \max\{\max\{\gamma(a), \gamma(a), \gamma(a)\}, \max\{\lambda(a), \lambda(a), \lambda(a)\}\} = (\gamma \vee \lambda)(a)$. This implies that $\gamma \vee \lambda \succeq \gamma \circ \lambda$. Therefore, $\tilde{g_\gamma} \Cap \tilde{f_\lambda} \ll \tilde{g_\gamma} \otimes \tilde{f_\lambda}$.

(2) $\Rightarrow$ (1). Let D and C be a $(2,1)$-ideal and a $(1,2)$-ideal of S, respectively. Then, by Theorem 2.2, $\chi_D\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_C\left(\tilde{S}_{\mathbf{S}}\right)$ are a hybrid $(2,1)$-ideal and a hybrid $(1,2)$-ideal in S over U, respectively. By assumption, we have $\chi_D\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_C\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_D\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_C\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{D\cap C}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(DC]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $D \cap C \subseteq (DC]$. Thus, by Lemma 3.10, S is completely regular. □

An ordered semigroup S satisfies $C15$ if for each $a \in S$ there exists $x \in S$ such that $a \le axa^2$. The following lemma is generalization of Theorem 3.16 in [13].

Lemma 3.11. *Let S be an ordered semigroup. Then the following statements are equivalent:*
(1) *S satisfies $C15$;*
(2) $C \cap O \subseteq (OC]$ *for every* $(1,2)$*-ideals* C *and* O *of* S*.*

Proof: This can be proved similarly as Theorem 3.16 in [13]. □

Lemma 3.11 can be used to prove the following theorem.

Theorem 3.11. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S satisfies $C15$;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid $(1,2)$-ideal $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ in S over U.*

Proof: (1) $\Rightarrow$ (2). Let $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ be a hybrid $(1,2)$-ideal in S over U. Let $a \in S$. Since S satisfies $C15$, there exists $x \in S$ such that $a \leq axa^2 \leq ax(axa^2)(axa^2) = (ax_1a^2)(axa^2)$ for some $x_1 \in S$. This implies that $S_a \neq \emptyset$. We obtain

$$\begin{aligned}\left(\tilde{g} \odot \tilde{f}\right)(a) &= \bigcup_{(u,v)\in S_a} \left[\tilde{g}(u) \cap \tilde{f}(v)\right] \\ &\supseteq \tilde{g}\left(ax_1a^2\right) \cap \tilde{f}\left(axa^2\right) \\ &\supseteq (\tilde{g}(a) \cap \tilde{g}(a) \cap \tilde{g}(a)) \cap \left(\tilde{f}(a) \cap \tilde{f}(a) \cap \tilde{f}(a)\right) \\ &= \left(\tilde{g} \cap \tilde{f}\right)(a).\end{aligned}$$

This implies that $\tilde{f} \cap \tilde{g} \sqsubseteq \tilde{g} \odot \tilde{f}$. Consider

$$\begin{aligned}(\gamma \circ \lambda)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{\gamma(u), \lambda(v)\}\} \\ &\leq \max\left\{\gamma\left(ax_1a^2\right), \lambda\left(axa^2\right)\right\} \\ &\leq \max\{\max\{\gamma(a), \gamma(a), \gamma(a)\}, \max\{\lambda(a), \lambda(a), \lambda(a)\}\} \\ &= (\gamma \vee \lambda)(a).\end{aligned}$$

This implies that $\gamma \vee \lambda \succeq \gamma \circ \lambda$. Therefore, $\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$.

(2) $\Rightarrow$ (1). Let C and O be a $(1,2)$-ideal of S. Then, by Theorem 2.2, $\chi_C\left(\tilde{S}_{\mathbf{S}}\right)$ and $\chi_O\left(\tilde{S}_{\mathbf{S}}\right)$ are hybrid $(1,2)$-ideals in S over U. By assumption, we have $\chi_O\left(\tilde{S}_{\mathbf{S}}\right) \Cap \chi_C\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_O\left(\tilde{S}_{\mathbf{S}}\right) \otimes \chi_C\left(\tilde{S}_{\mathbf{S}}\right)$. Then, by Lemma 2.2, we have $\chi_{O\cap C}\left(\tilde{S}_{\mathbf{S}}\right) \ll \chi_{(OC]}\left(\tilde{S}_{\mathbf{S}}\right)$. By Lemma 2.3, $O \cap C \subseteq (OC]$. By Lemma 3.10, S satisfies $C9$. $\square$

An ordered semigroup S satisfies $C16$ if for each $a \in S$ there exists $x \in S$ such that $a \leq a^2xa$. Similarly as Theorem 3.11, we have the following lemma.

Lemma 3.12. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S satisfies $C16$;*

(2) *$D \cap K \subseteq (DK]$ for every $(2,1)$-ideals D and K of S.*

In the same way, as in Theorem 3.11, we get the following result.

Theorem 3.12. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S satisfies $C16$;*

(2) *$\tilde{g}_\gamma \Cap \tilde{f}_\lambda \ll \tilde{g}_\gamma \otimes \tilde{f}_\lambda$ for every hybrid $(2,1)$-ideal $\tilde{g}_\gamma$ and $\tilde{f}_\lambda$ in S over U.*

We leave this section with an example of a hybrid (m,n)-ideal in an ordered semigroup. In ordered semigroups, this example demonstrates the advantages of hybrid ideals and how they differ from int-soft ideals and anti-fuzzy ideals.

Example 3.1. *Let $S = \{a,b,c,d,e\}$. Define a binary operation $\cdot$ and a partial relation $\leq$ on S shown as follows.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	d
c	a	a	a	a	d
d	d	d	d	d	d
e	e	e	e	e	e

and $\leq := \{(a, d)\} \cup \Delta_S$, *where* Δ_S *is the equality relation defined on* S. *Then,* S *is an ordered semigroup. Let* $U = \{u_1, u_2, u_3, u_4\}$. *Define a hybrid structure* $\tilde{f}_\lambda$ *in* S *over* U *by*

x	$\tilde{f}(x)$	$\lambda(x)$
a	U	0.1
b	$\{u_1\}$	0.8
c	$\{u_1, u_4\}$	0.7
d	$\{u_1, u_2, u_3\}$	0.4
e	$\{u_3\}$	0.9

We can carefully calculate that $\tilde{f}_\lambda$ *is a hybrid* $(2, 2)$*-ideal in* S *over* U, *but it is not a hybrid* $(1, 1)$*-ideal in* S *over* U. *Moreover, if we correctly consider* $\tilde{f}$, *then we can see that* $\tilde{f}$ *is an int-soft* $(2, 2)$*-ideal in ordered semigroup* S, *but it is not an int-soft* $(1, 1)$*-ideal. Similarly, we have that* λ *is an anti-fuzzy* $(2, 2)$*-ideal in ordered semigroup* S, *but it is not an anti-fuzzy* $(1, 1)$*-ideal.*

4. **Conclusions.** Classifications of regularities of ordered semigroups were studied over the years. Many researchers investigated characterizations of regularities by using many types of ideals. To study in such scope, we apply the concept of hybrid structures to classifing regularities of ordered semigroups. Future work is to study the concept of hybrid α-ideals of ordered semigroups and classify regularities by hybrid α-ideal of ordered semigroups.

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