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ON IDEAL ELEMENTS OF PARTIALLY ORDERED SEMIGROUPS WITH THE GREATEST ELEMENT

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Abstract. *In this paper, we study an algebraic structure, so-called poe-semigroups. Since every ordered semigroup embeds in a poe-semigroup, this algebraic structure is crucial for understanding ordered semigroups. The notion that plays a vital part in investigating poe-semigroups is ideal elements. It was known that (m, n)-ideal elements in poe-semigroups are a generalization of bi-ideal elements. We introduce the notion of n-interior ideal elements in poe-semigroups as a generalization of interior ideal elements. We use some combinations of (m, n)-ideal elements and n-interior ideal elements to characterize several classes of poe-semigroups. Moreover, we apply our results to hypersemigroups.*
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1. **Introduction.** Marty [36] introduced the concept of hyperalgebras in 1934. This concept is a generalization of the classical algebras in the sense that the composition of any two elements is a nonempty set instead of an element. The author considered group-like hyperalgebras, so-called hypergroups. Following this introduction, various authors investigated some generalizations of groups at the hyperalgebras level, both theoretical and practical implications (see [3, 7, 8, 37, 47]).

Bonansinga and Corsini [3] first studied the concept of hypersemigroups (semihypergroups or multisemigroups) in detail. Hypersemigroups have been researched in various ways by several researchers. Green's relations in hypersemigroups were studied by Hasankhani in [10]. Characterization of hypersemigroups into classes using various kinds of hyperideals and their generalizations have been investigated widely (see [6, 12, 23, 24]). Lekkoksung [33] investigated intuitionistic fuzzification version of bi-hyperideals of hypersemigroups in 2012. Hila and Naka [13] studied the purity of hyperradical in semihypergroups. The concept of basis in hypersemigroups was considered in [46] by Udom et al.

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A partially ordered semigroup (ordered semigroup or *po*-semigroup) is an algebraic system initially mentioned by Birkhoff in [1, 2]. One obtains new algebraic systems if *po*-semigroups satisfy some special properties. For example, a *po*-semigroup with the greatest element e is called a *poe*-semigroup. A *poe*-semigroup, which is also a joint-semilattice, is called a $\vee e$-semigroup. Moreover, a *poe*-semigroup, which is also a lattice, is called an *le*-semigroup (see [1, 2, 9]).

Fundamental ideal elements of *poe*-semigroups: left and right ideal elements were studied by Birkhoff (see [2]). Kehayopulu [16] explored some interesting properties of left and right ideal elements in some classes of $\vee e$-semigroups. The semiprime property of *poe*-semigroups was introduced by the author. Various ideal elements and special classes of *poe*-semigroups were introduced and studied by Kehayopulu in several directions (see [14, 15, 16, 17]). Kehayopulu [18, 19] investigated and studied Green's relations for *poe*-semigroups. Petro and Pasku [40] pointed out that Green's relations in *poe*-semigroups, so-called Green-Kehayopulu relations, behave differently from the original, especially for the $\mathcal{H}$-class. The $\mathcal{B}$- and $\mathcal{J}$-class in *poe*-semigroups have been also studied in [39, 43]. Any *poe*-semigroup is a *po*-semigroup, but certain researches have shown that any *po*-semigroup can be embedded into a *poe*-semigroup (see [20, 28, 29, 30]).

In 2015, Kehayopulu [21] defined the associativity of hyperoperation differently from the given by Marty in [36]. The author suggested that the associativity of hyperoperation should be redefined for reasonability. Kehayopulu [22] firstly discovered a link between *poe*-semigroups and semigroups after this study. The relationship between *poe*-semigroups, hypersemigroups, and other hyperalgebraic systems was then investigated (see [23, 24, 25, 26, 27]).

According to earlier research, *poe*-semigroups have specific characteristics that set them apart from *po*-semigroups. The representation of *po*-semigroups demonstrates the importance of *poe*-semigroups. Moreover, because of the relevance of studying *poe*-semigroups, particularly the relationship that *poe*-semigroups have with the study of hypersemigroups and other algebraic systems, we will concentrate on *poe*-semigroups in this paper. One of the essential concepts in the subject is the idea of ideal elements in *poe*-semigroups. Since the research of Kehayopulu, several ideal elements conceptions have been proposed. This study presents a new generalized notion of ideal elements in *poe*-semigroups. We define the concept of n-interior ideal elements in *poe*-semigroups and describe several classes of *poe*-semigroups by combining the ideas of n-interior ideal elements and (m, n)-ideal elements in Section 2 and Section 3, respectively. In Section 4, we apply our results to hypersemigroups. Then, several classes of hypersemigroups are characterized.

2. **Preliminaries.** A *po-semigroup* (*partially ordered semigroup or ordered semigroup*) is an algebraic system $\langle S; \cdot, \leq \rangle$ of type $(2; 2)$ consisting of a nonempty set S, a binary associative operation $\cdot$ on S and a partial relation $\leq$ on S such that for any $x, y \in S$

$$x \leq y \text{ implies } c \cdot x \leq c \cdot y \text{ and } x \cdot c \leq y \cdot c \qquad \text{(compatibility)}$$

for all $c \in S$. For the simplicity, the product of any elements a and b of S under the operation $\cdot$ is denoted by ab. For any $n \in \mathbb{N}$ and $a \in S$, the notation a^n stands for $a \cdots a$ the n-product of a. For the convenience, $a^0 b = b = ba^0$ for any $a, b \in S$.

Because any semigroup can be considered a *po*-semigroup, the concept of *po*-semigroups is a generalization of semigroups. A *poe-semigroup* is a *po*-semigroup having the greatest element $e \in S$, that is, $a \leq e$ for all $a \in S$. In a *poe*-semigroup $\langle S; \cdot, \leq \rangle$, we observe that

(1) for any $l, k \in \mathbb{N}$ such that $l \leq k$, we have that $e^k \leq e^l$;
(2) any element $a \in S$ and $m \in \mathbb{N} \setminus \{1\}$, $a \leq a^m$ implies $a \leq a^{km-k+1}$ for all $k \in \mathbb{N}$ (see [45]).

A $\vee e$-*semigroup* is an algebra $\langle S; \cdot, \vee\rangle$ of type $(2, 2)$ such that

(1) $\langle S; \cdot\rangle$ is a semigroup;
(2) $\langle S; \vee\rangle$ is a joint-semilattice with the greatest element $e \in S$;
(3) the operation $\cdot$ is distributive over $\vee$, that is, the identities $a(b \vee c) \approx ab \vee ac$ and $(a \vee b)c \approx ac \vee bc$ hold.

An *le-semigroup* is an algebra $\langle S; \cdot, \vee, \wedge\rangle$ of type $(2, 2, 2)$ such that

(1) $\langle S; \cdot\rangle$ is a semigroup;
(2) $\langle S; \vee, \wedge\rangle$ is a lattice with the greatest element $e \in S$;
(3) the operation $\cdot$ is distributive over $\vee$.

By the above definitions, we can see that any $\vee e$-semigroup $\langle S; \cdot, \vee\rangle$ can be considered as an algebraic system $\langle S; \cdot, \leq\rangle$ by assigning $a \leq b$ if and only if $a \vee b = b$. Similarly, any *le*-semigroup $\langle S; \cdot, \vee, \wedge\rangle$ can be regarded as an algebraic system $\langle S; \cdot, \leq\rangle$ by defining $a \leq b$ if and only if $a \vee b = b$ and $a \wedge b = a$. By the definition of $\leq$ given in this discussion, the distributivity of the operation $\cdot$ over $\vee$ implies the compatibility property. Therefore, we can consider any $\vee e$-semigroup and any *le*-semigroup as a *poe*-semigroup $\langle S; \cdot, \leq\rangle$. The readers can find more information about the stated algebraic systems in [1, 2, 9, 15, 16, 20]. From now on, we denote any *poe*-semigroup $\langle S; \cdot, \leq\rangle$ system by $\mathbf{S}$ the bold letter of its universe set.

Phochai and Changphas divided the regularities of *po*-semigroups into sixteen types in [41]. Several authors characterized some classes of *po*-semigroups through various kinds of ideals. For example, Cao [4] characterized regular ordered semigroups by left ideals, right ideals, and quasi-ideals. In 2006, Lee and Lee [32] provided descriptions of intra-regular ordered semigroups in terms of left (right) ideals and bi-ideals. We can reformulate these regularities in terms of *poe*-semigroups as follows. A *poe*-semigroup $\mathbf{S}$ is said to be

(P1) if for all $a \in S$, we have $a \leq eae$;
(P2) if for all $a \in S$, we have $a \leq ea$;
(P3) if for all $a \in S$, we have $a \leq ae$;
(P4) if for all $a \in S$, we have $a \leq eaeae$;
(P5) if for all $a \in S$, we have $a \leq eaea$;
(P6) if for all $a \in S$, we have $a \leq aeae$;
(P7) if for all $a \in S$, we have $a \leq aea$;
(P8) of degree n if for all $a \in S$, we have $a \leq ea^ne$ for some $n \in \mathbb{N} \setminus \{1\}$;
(P9) of degree n if for all $a \in S$, we have $a \leq ea^nea$ for some $n \in \mathbb{N} \setminus \{1\}$;
(P10) of degree n if for all $a \in S$, we have $a \leq aea^ne$ for some $n \in \mathbb{N} \setminus \{1\}$;
(P11) of degree n if for all $a \in S$, we have $a \leq aea^nea$ for some $n \in \mathbb{N} \setminus \{1\}$;
(P12) if for all $a \in S$, we have $a \leq ea^2$;
(P13) if for all $a \in S$, we have $a \leq a^2e$;
(P14) if for all $a \in S$, we have $a \leq a^2ea^2$;
(P15) if for all $a \in S$, we have $a \leq aea^2$;
(P16) if for all $a \in S$, we have $a \leq a^2ea$.

We can see that any intra-regular *poe*-semigroup [16] is a *poe*-semigroup satisfying (P8) of degree 2. Any (m, n)-regular *poe*-semigroup [17], where $m, n = 1$ is a *poe*-semigroup satisfying (P7). Moreover, any (m, n)-regular *poe*-semigroup, where $m, n > 1$ is a *poe*-semigroup satisfying (P14) and vice versa.

An element $a \in S$ of a *poe*-semigroup $\mathbf{S}$ is said to be

(1) a *subidempotent element* [2] of $\mathbf{S}$ if $a^2 \leq a$;
(2) an (m, n)-*ideal element* [17] of $\mathbf{S}$ if $a^mea^n \leq a$, where $m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The least (m,n)-ideal element of a $\vee e$-semigroup $\mathbf{S}$ greater than $a \in S$ is denoted by $\mathrm{I}_{(m,n)}(a)$. Kehayopulu [17] illustrated that $\mathrm{I}_{(m,n)}(a) = a \vee a^m e a^n$.

In [45], the concept of n-interior ideals in *po*-semigroups was introduced. Tiprachot et al. characterized all classes of *po*-semigroups using n-interior ideals. The readers may be found in [44] for more interesting ideals in *po*-semigroups that are the prevailing trends. In this paper, we define a similar notion introduced in [45] for *poe*-semigroups. We define the concept of n-interior ideal elements, which plays an essential role in this research.

Definition 2.1. *Let* $\mathbf{S}$ *be a poe-semigroup and* $n \in \mathbb{N}$*. An element* $a \in S$ *is an* n-interior ideal element *of* $\mathbf{S}$ *if it is subidempotent and* $ea^n e \leq a$*.*

Remark 2.1.

(1) For any $m, n \in \mathbb{N}$ *such that* $m < n$*, any* m*-interior ideal element is an* n*-interior ideal element. The converse is not true in general.*

(2) We can observe that a poe-semigroup's n*-interior ideal element is an element, whereas a po-semigroup's* n*-interior ideal is a set. This insight exemplifies the distinction between interior ideal elements and interior ideals.*

For the convenience, we illustrate an example of n-interior ideal elements in Section 4. The following result is not difficult to verify.

Proposition 2.1. *Let* $\mathbf{S}$ *be a poe-semigroup. Then*

(1) for any n*-interior ideal elements* a *and* b *of* $\mathbf{S}$ *if* $a \wedge b$ *exists, then* $a \wedge b$ *is an* n*-interior ideal element of* $\mathbf{S}$*;*

(2) for any $k \in \mathbb{N}$*, if* a *is an* n*-interior ideal element of* $\mathbf{S}$*, then* a^k *is also an* n*-interior ideal element of* $\mathbf{S}$*.*

Let $\mathbf{S}$ be a $\vee e$-semigroup. We denote the least n-interior ideal element of $\mathbf{S}$ greater than $a \in S$ by $\mathrm{I}_n(a)$. Then we obtain the following

Lemma 2.1. *Let* $\mathbf{S}$ *be a* $\vee e$*-semigroup and* $a \in S$*. Then*

$$\mathrm{I}_n(a) = a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e.$$

Proof: For our convenience in the proof, we suppose that $x = a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e$. Consider

$$\begin{aligned}
x^2 &= \left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e\right)\left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e\right) \\
&= \left[a^2 \vee \cdots \vee a^{n+1}\right] \vee \left[a^{n+2} \vee a^{n+3} \vee \cdots \vee a^{2n+2}\right] \vee \left[(ea^n e)x\right] \vee \left[x(ea^n e)\right] \\
&\leq \left[a^2 \vee \cdots \vee a^{n+1}\right] \vee \left[ea^n e\right] \vee \left[ea^n e\right] \vee \left[ea^n e\right] \\
&\leq a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e \\
&= x.
\end{aligned}$$

This shows that $x = a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e$ is a subidempotent element of $\mathbf{S}$.

Now, we consider

$$\begin{aligned}
ex^n e &= e\left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e\right)^n e \\
&= ea^n e \vee \cdots \vee ea^{n(n+1)} e \vee \left[\bigvee_{i,j \in \mathbb{N}_0}^{i+j=n-1} ex^i ea^n ex^j e\right] \\
&= ea^n e \leq a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e \\
&= x.
\end{aligned}$$

Assume that t is an n-interior ideal element of $\mathbf{S}$ such that $a \leq t$. Then

$$x = a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e \leq t \vee t^2 \vee \cdots \vee t^{n+1} \vee et^n e = t.$$

Altogether, we obtain that $a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e$ is the least n-interior ideal element of $\mathbf{S}$ greater than a. □

In the present paper, any (m, n)-ideal element is assumed to be subidempotent. Therefore, similarly to Lemma 2.1, it is not difficult to illustrate that $\mathrm{I}_{(m,n)}(a) = a \vee a^2 \vee \cdots \vee a^{m+n} \vee a^m ea^n$.

The representation of numerous classes of *poe*-semigroups was done in the next section using various types of (m, n)-ideal elements and n-interior ideal elements. The description of hypersemigroups is simply an example of the results shown in Section 3, as we will see in Section 4.

3. Characterizations of *poe*-semigroups.

This section provides characterizations of *poe*-semigroups using some combinations of (m, n)-ideal elements and n-interior ideal elements. Before we present our theorems, we note here that if a *poe*-semigroup $\mathbf{S}$ satisfies (Pk) for all $k \in \{1, \ldots, 16\}$, then we have $e^2 = e$.

Firstly, the class (P1) is described by 1-interior ideal elements.

Theorem 3.1. *Let* $\mathbf{S}$ *be a* $\vee e$*-semigroup. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *satisfies (P1);*

(2) $a \leq eae$ *for any* 1*-interior ideal element* a *of* $\mathbf{S}$.

Proof: (1) ⇒ (2). This direction is obvious.

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$a \leq \mathrm{I}_1(a) \leq e\,\mathrm{I}_1(a)e = e\left(a \vee a^2 \vee eae\right)e \leq eae.$$

This shows that $\mathbf{S}$ satisfies (P1). □

We can use $(0, 1)$-ideal elements to characterize the class (P2) as follows.

Theorem 3.2. *Let* $\mathbf{S}$ *be a* $\vee e$*-semigroup. Then the following conditions are equivalent:*

1. $\mathbf{S}$ *satisfies (P2);*
2. $a \leq ea$ *for any* $(0, 1)$*-ideal element* a *of* $\mathbf{S}$.

Proof: (1) ⇒ (2). This direction is obvious.

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$a \leq \mathrm{I}_{(0,1)}(a) \leq e\,\mathrm{I}_{(0,1)}(a) = e(a \vee ea) = ea.$$

This shows that $\mathbf{S}$ satisfies (P2). □

The following theorem can be proved similar to Theorem 3.2.

Theorem 3.3. *Let* $\mathbf{S}$ *be a* $\vee e$*-semigroup. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *satisfies (P3);*

(2) $a \leq ea$ *for any* $(1, 0)$*-ideal element* a *of* $\mathbf{S}$.

The combination between $(1, 1)$-ideals elements and 1-interior ideal elements is used to describe the class (P4).

Theorem 3.4. *Let* $\mathbf{S}$ *be an le-semigroup. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *satisfies (P4);*

(2) $a \wedge b \leq aba$ *for any* 1*-interior ideal element* a *and* $(1, 1)$*-ideal element* b *of* $\mathbf{S}$.

Proof: (1) ⇒ (2). Let a and b be a 1-interior ideal element and a $(1,1)$-ideal element of **S**, respectively. Since **S** satisfies (P4), we have $a = eae$. Then,

$$\begin{aligned} a \wedge b &\le e(a \wedge b)e(a \wedge b)e \\ &\le e(a \wedge b)e[e(a \wedge b)e(a \wedge b)e]e \\ &\le e(a)e[e(b)e(a)e]e \\ &= (eaee)b(eaee) \\ &= aba. \end{aligned}$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\le \mathrm{I}_1(a)\,\mathrm{I}_{(1,1)}(a)\,\mathrm{I}_1(a) \\ &= \big(a \vee a^2 \vee eae\big)\big(a \vee a^2 \vee aea\big)\big(a \vee a^2 \vee eae\big) \\ &= \big[a^3 \vee a^4 \vee a^5 \vee a^6\big] \vee \big[(eae)\,\mathrm{I}_{(1,1)}(a)\,\mathrm{I}_{(1)}(a)\big] \vee \big[\mathrm{I}_{(1)}(a)(aea)\,\mathrm{I}_{(1)}(a)\big] \\ &\quad \vee \big[\mathrm{I}_{(1)}(a)\,\mathrm{I}_{(1,1)}(a)(eae)\big] \\ &\le eaeae. \end{aligned}$$

This shows that **S** satisfies (P4). □

The concept of $(1,1)$-ideal elements and 1-interior ideal elements can also be applied to characterizing the class (P5).

Theorem 3.5. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P5);*
(2) $a \wedge b \le ab$ *for any* 1*-interior ideal element* a *and* $(1,1)$*-ideal element* b *of* **S**.

Proof: (1) ⇒ (2). Let a and b be a 1-interior ideal element and a $(1,1)$-ideal element of **S**, respectively. Since **S** satisfies (P5), we have $a = eae$. Then,

$$a \wedge b \le e(a \wedge b)e(a \wedge b) \le eaeb = ab.$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\le \mathrm{I}_1(a)\,\mathrm{I}_{(1,1)}(a) \\ &= \big(a \vee a^2 \vee eae\big)\big(a \vee a^2 \vee aea\big) \\ &= \big[a^2 \vee a^3 \vee a^4\big] \vee \big[(eae)\,\mathrm{I}_{(1,1)}(a)\big] \vee [\mathrm{I}_1(a)(aea)] \\ &\le eaea. \end{aligned}$$

This shows that **S** satisfies (P5). □

We can characterize an *le*-semigroup satisfying (P6) similar to Theorem 3.5.

Theorem 3.6. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P6);*
(2) $a \wedge b \le ab$ *for any* $(1,1)$*-ideal element* a *and* 1*-interior ideal element* b *of* **S**.

The following equivalence is obtained with the help of $(1,0)$-ideal elements, $(0,1)$-ideal elements, and 1-interior ideal elements.

Theorem 3.7. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P7);*
(2) $a \wedge b \wedge c \le abc$ *for any* $(1,0)$*-ideal element* a*,* 1*-interior ideal element* b *and* $(0,1)$*-ideal element* c *of* **S**.

Proof: (1) ⇒ (2). Let a, b and c be a $(1,0)$-ideal element, an n-interior ideal element and a $(0,1)$-ideal element of $\mathbf{S}$. Since $\mathbf{S}$ satisfies (P7), we have $a = ae$ and $c = ec$. Then,

$$\begin{aligned} a \wedge b \wedge c &\leq (a \wedge b \wedge c)e(a \wedge b \wedge c) \\ &\leq [(a \wedge b \wedge c)e(a \wedge b \wedge c)]\, e\, [(a \wedge b \wedge c)e(a \wedge b \wedge c)] \\ &\leq (a \wedge b \wedge c)e(a \wedge b \wedge c)e(a \wedge b \wedge c) \\ &\leq aebec \\ &= abc. \end{aligned}$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_{(1,0)}(a)\, \mathrm{I}_1(a)\, \mathrm{I}_{(0,1)}(a) \\ &= \left[a^3 \vee a^4\right] \vee \left[(ae)\, \mathrm{I}_1(a)\, \mathrm{I}_{(0,1)}(a)\right] \vee \left[\mathrm{I}_{(1,0)}(a)(eae)\, \mathrm{I}_{(0,1)}(a)\right] \vee \left[\mathrm{I}_{(1,0)}(a)\, \mathrm{I}_1(a)(ea)\right] \\ &\leq (aea) \vee (aea) \vee (aea) \vee (aea) \\ &= aea. \end{aligned}$$

This shows that $\mathbf{S}$ satisfies (P7). □

The notion of n-interior ideal elements represents the following class of *poe*-semigroups.

Theorem 3.8. *Let* $\mathbf{S}$ *be an le-semigroup and* $n \in \mathbb{N} \setminus \{1\}$*. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *satisfies (P8) of degree* n*;*
(2) $a \wedge b \leq ab$ *for any* n*-interior ideal elements* a *and* b *of* $\mathbf{S}$*.*

Proof: (1) ⇒ (2). Let a and b be n-interior ideal elements of $\mathbf{S}$. Since $\mathbf{S}$ satisfies (P8) of degree n, we have $a = ea^ne$ and $b = eb^ne$. Then,

$$\begin{aligned} a \wedge b &\leq e(a \wedge b)^n e \\ &\leq e\left[e(a \wedge b)^n e\right]^n e \\ &= e\left[e(a \wedge b)^n e\right]\left[e(a \wedge b)^n e\right]\left[e(a \wedge b)^n e\right]^{n-2} e \\ &\leq \left[e(a \wedge b)^n e\right]\left[e(a \wedge b)^n e\right] \\ &\leq (ea^ne)(eb^ne) \\ &= ab. \end{aligned}$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_n(a)\, \mathrm{I}_n(a) \\ &= \left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^ne\right)\left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^ne\right) \\ &= \left[a^2 \vee \cdots \vee a^{n+1}\right] \vee \left[a^{n+2} \vee a^{n+3} \vee \cdots \vee a^{2n+2}\right] \vee \left[(ea^ne)\, \mathrm{I}_n(a)\right] \vee \left[\mathrm{I}_n(a)(ea^ne)\right] \\ &\leq ea^ne \vee ea^ne \vee ea^ne \vee ea^ne \\ &= ea^ne. \end{aligned}$$

This shows that $\mathbf{S}$ satisfies (P8) of degree n. □

The notion of n-interior ideal elements together with $(0,1)$-ideal elements can be used to illustrate the class (P9) as shown below.

Theorem 3.9. *Let* $\mathbf{S}$ *be an le-semigroup and* $n \in \mathbb{N} \setminus \{1\}$*. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *satisfies (P9) of degree* n*;*
(2) $a \wedge b \leq ab$ *for any* n*-interior ideal element* a *and* $(0,1)$*-ideal element* b *of* $\mathbf{S}$*.*

Proof: (1) $\Rightarrow$ (2). Let a and b be an n-interior ideal element and a $(0,1)$-ideal element of $\mathbf{S}$, respectively. Since $\mathbf{S}$ satisfies (P9) of degree n, we have $a = ea^n e$. Then,

$$a \wedge b \leq e(a \wedge b)^n e(a \wedge b) \leq ea^n eb = ea^n eeb \leq ab.$$

(2) $\Rightarrow$ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_n(a)\, \mathrm{I}_{(0,1)}(a) \\ &= \left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e\right)(a \vee ea) \\ &= \left[a^2 \vee \cdots \vee a^{n+2}\right] \vee \left[(ea^n e)\, \mathrm{I}_{(0,1)}(a)\right] \vee \left[\mathrm{I}_n(a)(ea)\right] \\ &\leq ea^n ea. \end{aligned}$$

This shows that $\mathbf{S}$ satisfies (P9) of degree n. □

Similarly, we obtain the following theorem.

Theorem 3.10. *Let* $\mathbf{S}$ *be an le-semigroup and* $n \in \mathbb{N} \setminus \{1\}$*. Then the following conditions are equivalent:*
(1) $\mathbf{S}$ *satisfies (P10) of degree* n*;*
(2) $a \wedge b \leq ab$ *for any* $(1,0)$*-ideal element* a *and* n*-ideal element* b *of* $\mathbf{S}$*.*

The class (P11) is described by the three varieties of (m,n)-ideal elements shown as follows.

Theorem 3.11. *Let* $\mathbf{S}$ *be an le-semigroup and* $n \in \mathbb{N} \setminus \{1\}$*. Then the following conditions are equivalent:*
(1) $\mathbf{S}$ *satisfies (P11) of degree* n*;*
(2) $a \wedge b \wedge c \leq abc$ *for any* $(1,0)$*-ideal element* a*,* n*-ideal element* b *and* $(0,1)$*-ideal element* c *of* $\mathbf{S}$*.*

Proof: (1) $\Rightarrow$ (2). Let a, b and c be a $(1,0)$-ideal element, an n-interior ideal element and a $(0,1)$-ideal element of $\mathbf{S}$, respectively. Since $\mathbf{S}$ satisfies (P11) of degree n, we have $b = eb^n e$. Then,

$$a \wedge b \wedge c \leq (a \wedge b \wedge c)e(a \wedge b \wedge c)^n e(a \wedge b \wedge c) \leq a(eb^n e)c = abc.$$

(2) $\Rightarrow$ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_{(1,0)}(a)\, \mathrm{I}_n(a)\, \mathrm{I}_{(0,1)}(a) \\ &= (a \vee ae)\left(a \vee a^2 \vee \cdots \vee a^{n+1} \vee ea^n e\right)(a \vee ea) \\ &= \left[a^3 \vee \cdots \vee a^{n+3}\right] \vee \left[(ea)\, \mathrm{I}_1(a)\, \mathrm{I}_{(0,1)}(a)\right] \vee \left[\mathrm{I}_{(1,0)}(a)\, (ea^n e)\, \mathrm{I}_{(0,1)}(a)\right] \vee \left[\mathrm{I}_{(1,0)}(a)\, \mathrm{I}_1(a)(ea)\right] \\ &\leq aea^n ea. \end{aligned}$$

This shows that $\mathbf{S}$ satisfies (P11) of degree n. □

The following result applies the $(0,2)$-ideal elements in characterizing the class (P12).

Theorem 3.12. *Let* $\mathbf{S}$ *be a* $\vee e$*-semigroup. Then the following conditions are equivalent:*
(1) $\mathbf{S}$ *satisfies (P12);*
(2) $a \leq ea^2$ *for any* $(0,2)$*-ideal element* a *of* $\mathbf{S}$*.*

Proof: (1) $\Rightarrow$ (2). This direction is obvious.

(2) $\Rightarrow$ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_{(0,2)}(a) \\ &\leq e\, \mathrm{I}_{(0,2)}(a)\, \mathrm{I}_{(0,2)}(a) \\ &= e\left(\left[a^2 \vee a^3 \vee a^4\right] \vee \left[(ea^2)\, \mathrm{I}_{(0,2)}(a)\right] \vee \left[\mathrm{I}_{(0,2)}(a)(ea^2)\right]\right) \end{aligned}$$

$$\leq e(ea^2)$$
$$\leq ea^2.$$

This shows that **S** satisfies (P12). □

By the above theorem, we obtain Theorem 3.13 by applying similar arguments.

Theorem 3.13. *Let* **S** *be a* $\vee e$*-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P13);*
(2) $a \leq a^2e$ *for any* $(2,0)$*-ideal element* a *of* **S**.

Combining $(2,0)$-ideal elements and $(0,2)$-ideal elements, we obtain the following characterization.

Theorem 3.14. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P14);*
(2) $a \wedge b \leq ab$ *for any* $(2,0)$*-ideal element* a *and* $(0,2)$*-ideal element* b *of* **S**.

Proof: (1) ⇒ (2). Let a and b be a $(2,0)$-ideal element and a $(0,2)$-ideal element of **S**, respectively. Since **S** satisfies (P14), we have $a = a^2e$ and $b = eb^2$. Then,

$$a \wedge b \leq (a \wedge b)^2 e (a \wedge b)^2 \leq a^2eb^2 = (a^2e)(eb^2) = ab.$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_{(2,0)}(a)\,\mathrm{I}_{(0,2)}(a) \\ &= \left(a \vee a^2 \vee a^2e\right)\left(a \vee a^2 \vee ea^2\right) \\ &= \left[a^2 \vee a^3 \vee a^4\right] \vee \left[\left(a^2e\right)\mathrm{I}_{(0,2)}(a)\right] \vee \left[\mathrm{I}_{(2,0)}(a)\left(ea^2\right)\right] \\ &\leq a^2ea^2. \end{aligned}$$

This shows that **S** satisfies (P14). □

The class (P15) can be described using $(1,0)$-ideal elements and $(0,2)$-ideal elements as follows.

Theorem 3.15. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P15);*
(2) $a \wedge b \leq ab$ *for any* $(1,0)$*-ideal element* a *and* $(0,2)$*-ideal element* b *of* **S**.

Proof: (1) ⇒ (2). Let a and b be a $(1,0)$-ideal element and a $(0,2)$-ideal element of **S**, respectively. Since **S** satisfies (P15), we have $a = ae$ and $b = eb^2$. Then,

$$a \wedge b \leq (a \wedge b)e(a \wedge b)^2 \leq aeb^2 = (ae)\left(eb^2\right) = ab.$$

(2) ⇒ (1). Let $a \in S$. By our assumption, we have

$$\begin{aligned} a &\leq \mathrm{I}_{(1,0)}(a)\,\mathrm{I}_{(0,2)}(a) \\ &= (a \vee ae)\left(a \vee a^2 \vee ea^2\right) \\ &= \left[a^2 \vee a^3\right] \vee \left[(ae)\,\mathrm{I}_{(0,2)}(a)\right] \vee \left[\mathrm{I}_{(1,0)}(a)\left(ea^2\right)\right] \\ &\leq aea^2. \end{aligned}$$

This shows that **S** satisfies (P15). □

Similarly to Theorem 3.15, we obtain the following result.

Theorem 3.16. *Let* **S** *be an le-semigroup. Then the following conditions are equivalent:*
(1) **S** *satisfies (P16);*
(2) $a \wedge b \leq ab$ *for any* $(2,0)$*-ideal element* a *and* $(0,1)$*-ideal element* b *of* **S**.

4. **Applications of the Characterizations.** In our last section, we present connections between *poe*-semigroups and hypersemigroups. Firstly, we recall some terminologies of hypersemigroups. Hypersemigroups are sometimes referred to as semihypergroups or multisemigroups in some literature (see [3, 31]).

A (*binary*) *hyperoperation* $\circ$ on a nonempty set H is a mapping $\circ\colon H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all subsets of H without empty set. A *hypergroupoid* is a structure $\langle H; \circ\rangle$ comprising a nonempty set H and a hyperoperation defined on it. For any hypergroupoid $\langle H; \circ\rangle$, the hyperoperation $\circ$ induces a mapping $*\colon \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H)$ defined by

$$A * B := \bigcup_{a\in A, b\in B} (a \circ b) \tag{1}$$

for all $A, B \in \mathcal{P}^*(H)$ (see [21]).

A hypergroupoid $\langle H; \circ\rangle$ is said to be a *hypersemigroup* if

$$\{a\} * (b \circ c) = (a \circ b) * \{c\},$$

equivalently,

$$\{a\} * (\{b\} * \{c\}) = (\{a\} * \{b\}) * \{c\}$$

for any $a, b, c \in H$ (see [24, Proposition 4]).

Remark 4.1. *We can see that we cannot define the associative property in hypersemigroup using only the hyperoperation $\circ$. In fact, if we define the associativity of hypergroupoid by $a \circ (b \circ b) = (a \circ b) \circ c$, then we can ask for the meanings of $a \circ (b \circ c)$ and $(a \circ b) \circ c$. That is, $a \circ (b \circ c)$ means that the "element" a* hyperoperates *with the "set" $(b \circ c)$. This does not make sense in the definition of the hyperoperation $\circ$.*

Kehayopulu was illustrated the interconnections between hypersemigroup $\langle H; \circ\rangle$ and the algebraic structure $\langle \mathcal{P}^*(H); *, \subseteq\rangle$ induced by the operation $\circ$.

Theorem 4.1. [25] *If $\langle H; \circ\rangle$ is a hypersemigroup, then $\langle \mathcal{P}^*(H); *, \subseteq\rangle$ is an lH-semigroup. In this case, for any $A, B \in \mathcal{P}^*(H)$, the* least upper bound *of A and B is $A \cup B$, and the* greatest lower bound *of A and B is $A \cap B$.*

However, there is an error in Theorem 4.1 as given by the following question: what is the greatest lower bound of any singleton sets? By this question, we need a little modification. For any hypersemigroup $\langle H; \circ\rangle$, the hyperoperation $\circ$ can be extended to a binary operation $\widehat{\circ}$ defined on $\mathcal{P}(H)$ the set of all subsets of H by

$$A \,\widehat{\circ}\, B := \begin{cases} \bigcup\limits_{a\in A, b\in B} (a \circ b) & \text{if } A, B \neq \emptyset, \\ \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset, \end{cases}$$

for all $A, B \in \mathcal{P}(H)$. By this setting, we obtain an algebraic system $\langle \mathcal{P}(H); \widehat{\circ}, \subseteq\rangle$. Since $\emptyset$ is the zero element of $\langle \mathcal{P}(H); \widehat{\circ}, \subseteq\rangle$ and the operation $\widehat{\circ}$ is defined in terms of $*$, by Theorem 4.1, we conclude that $\langle \mathcal{P}(H); \widehat{\circ}, \subseteq\rangle$ is a $\vee H$-semigroup. Moreover, the greatest lower bound of any two elements of $\mathcal{P}(H)$ exists. Therefore, we obtain the following result.

Theorem 4.2. *Let $\langle H; \circ\rangle$ be a hypersemigroup. Then $\langle \mathcal{P}(H); \widehat{\circ}, \subseteq\rangle$ is an lH-semigroup.*

From now on, we denote any hypersemigroup $\langle H; \circ\rangle$ by $\mathbf{H}$ the bold letter of its universe set. Furthermore, we write $\widehat{\mathbf{H}}$ as a notation of the algebraic structure $\langle \mathcal{P}(H); \widehat{\circ}, \subseteq\rangle$ induced by the hypersemigroup $\mathbf{H}$.

For any hypersemigroup $\mathbf{H}$ and $A, B \in \mathcal{P}(H)$, we write AB instead of $A \,\widehat{\circ}\, B$. In particular, if $A = \{a\}$, where $a \in H$, we write $\{a\}B$ and $B\{a\}$ by aB and Ba, respectively.

Moreover, we let $A^0B = B = BA^0$. By Theorem 4.2, we have that for any $A \in \mathcal{P}(H)$ and for $n \in \mathbb{N}$ the notion

$$\underbrace{A \,\widehat{\circ} \cdots \widehat{\circ}\, A}_{n \text{ terms}}$$

is meaningful, and we denote it by A^n.

The concept of (m, n)-hyperideals was first introduced in ordered hypersemigroups by Mahboob et al. (see [34]). However, by Remark 4.1, we can define this concept in hypersemigroups and redefine it as more consequential.

Definition 4.1. *A nonempty subset A of a hypersemigroup $\mathbf{H}$ is called*

*(1) an (m, n)-*hyperideal*, where $m, n \in \mathbb{N}_0$, of $\mathbf{H}$ if A is an (m, n)-ideal element of $\widehat{\mathbf{H}}$;*
*(2) an n-*interior hyperideal*, where $n \in \mathbb{N}_0$, of $\mathbf{H}$ if A is an n-interior ideal element of $\widehat{\mathbf{H}}$.*

We can see that any left hyperideal is a $(0, 1)$-ideal and any right hyperideal is a $(1, 0)$-hyperideal. Therefore, the notion of (m, n)-hyperideals is a generalization of that of left and right hyperideal (see [10]). Any interior hyperideal [11] is a 1-interior hyperideal. By Definition 4.1, the following observation can be obtained immediately.

Corollary 4.1. *Let $\mathbf{H}$ be a hypersemigroup, A is a nonempty subset of H and $m, n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) A is an (m, n)-hyperideal (resp., n-interior hyperideal) of $\mathbf{H}$;
(2) A is an (m, n)-ideal element (resp., n-interior ideal element) of $\widehat{\mathbf{H}}$.

Example 4.1. *Let $\langle S; \cdot, \leq\rangle$ be an ordered semigroup defined by Example 3.2 in* [45]. *The operation $\cdot$ and an order relation $\leq$ can be illustrated as follows,*

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	a	b
c	a	a	a	a	a	b
d	a	a	a	a	a	d
e	a	d	d	a	a	d
f	a	d	d	d	e	f

and $\leq := \Delta_S \cup \{(d, a)\}$, where $\Delta_S := \{(x, x) : x \in S\}$. By Ends lemma, we obtain that $\mathbf{S} := \langle S, \circ\rangle$ is a semihypergroup. Here $\circ$ is a hyperoperation defined by $a \circ b := (a \cdot b]_{\leq}$, where $(a \cdot b]_{\leq} := \{x \in S : a \cdot b \leq x\}$ for all $a, b \in S$ (see [5, 38]). *By our discussion, we obtain an lS-semigroup $\widehat{\mathbf{S}} := \langle \mathcal{P}(S); \widehat{\circ}, \subseteq\rangle$. Then, we can calculate that $\{a, b\}$ is a 2-interior ideal element of $\widehat{\mathbf{S}}$, but it is not an interior ideal element of $\widehat{\mathbf{S}}$. This demonstrates that the notion of n-interior ideal elements is a generalization of 1-interior ideal elements.*

Some classes of hypersemigroups: regular and intra-regular first appeared in [42] in terms of hyperoperation $\circ$. By Remark 4.1, we can redefine these classes in terms of $\widehat{\circ}$ as follows. A hypersemigroup $\mathbf{H}$ satisfies

(H1) if for all $a \in H$, we have $a \in HaH$,
(H2) if for all $a \in H$, we have $a \in Ha$,
(H3) if for all $a \in H$, we have $a \in aH$,
(H4) if for all $a \in H$, we have $a \in HaHaH$,
(H5) if for all $a \in H$, we have $a \in HaHa$,
(H6) if for all $a \in H$, we have $a \in aHaH$,
(H7) if for all $a \in H$, we have $a \in aHa$,

(H8) of degree n if for all $a \in H$, we have $a \in Ha^nH$ for some $n \in \mathbb{N} \setminus \{1\}$,
(H9) of degree n if for all $a \in H$, we have $a \in Ha^nHa$ for some $n \in \mathbb{N} \setminus \{1\}$,
(H10) of degree n if for all $a \in H$, we have $a \in aHa^nH$ for some $n \in \mathbb{N} \setminus \{1\}$,
(H11) of degree n if for all $a \in H$, we have $a \in aHa^nHa$ for some $n \in \mathbb{N} \setminus \{1\}$,
(H12) if for all $a \in H$, we have $a \in Ha^2$,
(H13) if for all $a \in H$, we have $a \in a^2H$,
(H14) if for all $a \in H$, we have $a \in a^2Ha^2$,
(H15) if for all $a \in H$, we have $a \in aHa^2$,
(H16) if for all $a \in H$, we have $a \in a^2Ha$.

Any regular hypersemigroup [35] is a hypersemigroup satisfying (H7). A hypersemigroup satisfying (H8) of degree 2 is an intra-regular hypersemigroup [35].

We obtain the following consequence immediately by simple observation.

Corollary 4.2. *Let* $\mathbf{H}$ *be a hypersemigroup and* $k \in \{1, \dots, 16\}$*. Then the following statements are equivalent:*

(1) $\mathbf{H}$ *satisfies (Hk);*
(2) $\widehat{\mathbf{H}}$ *satisfies (Pk).*

The following corollary provides an example of how our main results describe the classes of hypersemigroups. Only the characterization of hypersemigroups satisfying (H8) is presented. Other classes can be demonstrated in the same way.

Corollary 4.3. *Let* $\mathbf{H}$ *be a hypersemigroup and* $n \in \mathbb{N}$*. Then the following statements are equivalent:*

(1) $\mathbf{H}$ *satisfies (H8) of degree* n*;*
(2) $A \cap B \subseteq AB$ *for any* n*-interior hyperideals* A *and* B *of* $\mathbf{H}$*.*

Proof: (1) $\Rightarrow$ (2). Let A and B be n-interior hyperideals of $\mathbf{H}$. By Definition 4.1, A and B are n-interior ideal elements of $\widehat{\mathbf{H}}$. Since $\mathbf{H}$ satisfies (H8), by Corollary 4.2, we have that $\widehat{\mathbf{H}}$ satisfies (P8). By Theorem 3.8, we obtain the claim.

(2) $\Rightarrow$ (1). Let A and B be n-interior ideal elements of $\widehat{\mathbf{H}}$. By Definition 4.1, A and B are n-interior hyperideal of $\mathbf{H}$. By our presumption and Theorem 3.8, we have that $\widehat{\mathbf{H}}$ satisfies (P8). By Corollary 4.2, we obtain that $\mathbf{H}$ satisfies (H8). □

5. **Conclusions.** In this paper, we define the notion of n-interior ideals in *poe*-semigroups. We characterize *poe*-semigroups in terms of (m, n)-ideal elements and n-interior ideal elements. The characterizations can be applied to hypersemigroups. As a result, hypersemigroup new characterizations are also achieved. Our results impact not just hypersemigroups but also other algebraic structures such as semigroups, ordered hypersemigroups, fuzzy semigroups and fuzzy ordered semigroups (see [27, 30]). These illustrate the importance of studying *po*-semigroups with the greatest element. We remark that although many results in hypersemigroups and other algebraic systems may be seen through *po*-semigroups with the greatest element, hypersemigroups and other algebraic systems are nevertheless useful in real-world applications.

We end this paper with the following problems.

1) Are there any properties of hypersemigroups that can be studied in terms of *poe*-semigroups? Is it possible to study the radical ideal elements and the pure ideal elements?
2) Is it possible if hypersemirings can be investigated in this direction? In fact, can we define *poe*-semirings?

3) Can we use the fuzzification settings to investigate *po*-semigroups with the greatest element?

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