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ON BIPOLAR-VALUED SUBBISEMIRINGS OF BISEMIRINGS AND THEIR EXTENSION

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Abstract. *We defined bipolar-valued subbisemirings, level sets of bipolar-valued subbisemirings, and bipolar-valued normal subbisemirings of bisemirings. Additionally, we look into some of these subbisemirings related properties (shortly, SBS). Let A be a bipolar-valued fuzzy set (BVFS) in S. Prove that* $\tilde{f} = \langle f_A^p, f_A^n \rangle$ *is a bipolar-valued subbisemiring of S if and only if all non-empty level set* $\tilde{f}^{(t,s)}$ *is a subbisemiring of S for* $t \in [0, 1]$ *and* $s \in [-1, 0]$*. Let A be a BVSBS of a bisemiring S and V be the strongest bipolar-valued relation of S. Prove that A is a BVSBS of S if and only if V is a BVSBS of* $S \times S$*. The homomorphic image and pre-image of BVSBS are also BVSBS. Let* $f_{\tilde{\alpha}}$ *be an* (α, β)*-BVSBS of S. Prove that the nonempty sets* f_α^p *and* f_α^n *are SBSs of S, where* $f_\alpha^p = \{p \in S \mid f^p(p) > \alpha^p\}$ *and* $f_\alpha^n = \{p \in S \mid f^n(p) < \alpha^n\}$*. Let* $\tilde{f} = \langle f_A^p, f_A^n \rangle$ *be any BVFS in S. Prove that* $\tilde{f}$ *is an* (α, β)*-BVSBS of S if and only if each non-empty level subset* $\tilde{f}^{(t,s)}$ *is an SBS of S for all* $t \in (\alpha^p, \beta^p]$ *and* $s \in (\alpha^n, \beta^n]$*. Examples are given to demonstrate our findings.*

Keywords: Subbisemiring, Bipolar-valued subbisemiring, (α, β)-bipolar-valued subbisemiring, (α, β)-bipolar-valued normal subbisemiring, Homomorphism

1. **Introduction.** The various ideals based on semirings have been described by a number of authors and academics [1]. The German mathematician Dedekind initiated the study of semirings in relation to the ideals of commutative rings. The American mathematician Vandever later explored semirings and recognized them as a basic algebraic structure in 1934. It is a generalization of distributive lattices and rings. However, since 1950, there have been improvements in semiring theory. In 1965, Zadeh [2] introduced the fuzzy set theory. A bipolar fuzzy set is an extension of a fuzzy set in which membership degree range is $[-1, 1]$ [3]. The membership degree range of the bipolar fuzzy set is expanded from the interval $[0, 1]$ to $[-1, 0]$. The idea which lies behind such description is connected with the existence of bipolar information (positive information and negative information) about the given set. Information that would be positive indicates what is accepted as possible, whereas information that is negative shows what is thought to be absolutely impossible.

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In reality, a large number of human decisions are founded on dualistic or bipolar judgment thinking, which has both a positive and a negative side. For example, collaboration and competitiveness, hostile opposition, shared interests, effect and side effects, probability and conflict of interest improbability and other concepts are the two parties frequently collaborate. Lee [4] discussed the concept of BVFSs and their operations. Palanikumar and Arulmozhi [5, 6, 7, 8, 9] presented various fuzzy ideals of bisemirings and semigroups. A semiring $(S, +, \cdot)$ is a non-empty set in which $(S, +)$ and $(S, \cdot)$ are semigroups such that "$\cdot$" is distributive over "$+$". Ahsan et al. [10] presented the idea of fuzzy semirings in 1993. Sen and Ghosh [11] introduced the notion for bisemirings in 2001. A bisemiring $(S, +, \circ, \times)$ is an algebraic structure in which $(S, +, \circ)$ and $(S, \circ, \times)$ are semirings in which $(S, +), (S, \circ)$, and $(S, \times)$ are semigroups such that (1) $x \circ (y + z) = x \circ y + x \circ z$, (2) $(y + z) \circ x = y \circ x + z \circ x$, (3) $x \times (y \circ z) = (x \times y) \circ (x \times z)$, and (4) $(y \circ z) \times x = (y \times x) \circ (z \times x)$ for all $x, y, z \in S$. A non-empty subset A of a bisemiring $(S, +, \circ, \times)$ is an SBS of S if and only if $x + y \in A$, $x \circ y \in A$, and $x \times y \in A$ for all $x, y \in A$ [12]. Palanikumar et al. discussed various algebraic structures and its applications [13, 14, 15, 16, 17, 18, 19]. The goal of this study is to investigate and make conclusions on several aspects of the subbisemiring theory to BVSBS theory. The following five sections make up the article. Section 1 contains the introduction, and Section 2 has the semiring and SBS preliminary facts. The BVSBS hypothesis is contained in Section 3. In Section 4, the idea of (α, β)-BVSBS homomorphism is proposed, and its features are discussed. The theory of (α, β)-BVNSBS homomorphism is introduced in Section 5. Additionally, when evaluating the BVSBS and BVNSBS, use some numerical examples.

2. **Preliminaries.** In this section, we quickly recall some of the basic definitions required for our further studies.

Definition 2.1. [3] *Let $(S, +, \cdot)$ be a semiring. A fuzzy set A in S is said to be a fuzzy subsemiring of S if it satisfies the following conditions:*

(1) $f_A(x + y) \geq \min\{f_A(x), f_A(y)\}$,
(2) $f_A(x \cdot y) \geq \min\{f_A(x), f_A(y)\}$, $\forall x, y \in S$.

Definition 2.2. [4] *The BVFS A in a universe X is an object having the form $A = \{\langle x, f_A^p(x), f_A^n(x)\rangle \mid x \in X\}$, where $f_A^p\colon X \to [0, 1]$ and $f_A^n\colon X \to [-1, 0]$. Here $f_A^p(x)$ represents the degree of satisfaction of the element x to the property and $f_A^n(x)$ represents the degree of satisfaction of x to some implicit counter property of A. For simplicity, the symbol $\langle f_A^p, f_A^n\rangle$ is used for the BVFS $A = \{\langle x, f_A^p(x), f_A^n(x)\rangle \mid x \in X\}$.*

Definition 2.3. *Let $A = \langle f_A^p, f_A^n\rangle$ and $B = \langle f_B^p, f_B^n\rangle$ be two BVFSs in a non-empty set X. Then*

(1) $A \cap B = \{\langle x, \min\{f_A^p(x), f_B^p(x)\}, \max\{f_A^n(x), f_B^n(x)\}\rangle \mid x \in X\}$,
(2) $A \cup B = \{\langle x, \max\{f_A^p(x), f_B^p(x)\}, \min\{f_A^n(x), f_B^n(x)\}\rangle \mid x \in X\}$.

Definition 2.4. *For any BVFS $A = \langle f_A^p, f_A^n\rangle$ in a non-empty set X, we defined the (α, β)-cut of A as the crisp subset $\{x \in X \mid f_A^p(x) \geq \alpha$ and $f_A^n(x) \leq \beta\}$ of X.*

Definition 2.5. *Let A and B be fuzzy sets in S_1 and S_2, respectively. The product of A and B denoted by $A \times B$ is defined as $A \times B = \{f_{A\times B}(s_1, s_2) \mid s_1 \in S_1$ and $s_2 \in S_2\}$, where $f_{A\times B}(s_1, s_2) = \min\{f_A(s_1), f_B(s_2)\}$ for all $s_1 \in S_1$ and $s_2 \in S_2$.*

Definition 2.6. [5] *The fuzzy set A in a bisemiring $(S, \boxtimes_1, \boxtimes_2, \boxtimes_3)$ is said to be a fuzzy subbisemiring (FSBS) of S if it satisfies the following conditions:*

(1) $f_A(x \boxtimes_1 y) \geq \min\{f_A(x), f_A(y)\}$,

(2) $f_A(x \boxtimes_2 y) \geq \min\{f_A(x), f_A(y)\}$,
(3) $f_A(x \boxtimes_3 y) \geq \min\{f_A(x), f_A(y)\}$, $\forall x, y \in S$.

Definition 2.7. [5] *The FSBS A of a bisemiring* $(S, \boxtimes_1, \boxtimes_2, \boxtimes_3)$ *is said to be a fuzzy normal subbisemiring (FNSBS) of S if it satisfies the following conditions:*
(1) $f_A(x \boxtimes_1 y) = f_A(y \boxtimes_1 x)$,
(2) $f_A(x \boxtimes_2 y) = f_A(y \boxtimes_2 x)$,
(3) $f_A(x \boxtimes_3 y) = f_A(y \boxtimes_3 x)$, $\forall x, y \in S$.

Definition 2.8. [12] *Let* $(S, +, \cdot, \times)$ *and* $(T, \oplus, \circ, \otimes)$ *be two bisemirings. A function* ϕ: $S \to T$ *is said to be a homomorphism if it satisfies the following conditions:*
(1) $\phi(x + y) = \phi(x) \oplus \phi(y)$,
(2) $\phi(x \cdot y) = \phi(x) \circ \phi(y)$,
(3) $\phi(x \times y) = \phi(x) \otimes \phi(y)$, $\forall x, y \in S$.

3. **Bipolar-Valued Subbisemirings.** In what follows, let $S = (S, \boxtimes_1, \boxtimes_2, \boxtimes_3)$ denote a bisemiring unless otherwise stated.

Definition 3.1. *Let S be the SBS. The BVFS* $A = \langle f_A^p, f_A^n \rangle$ *in S is said to be a bipolar-valued subbisemiring (BVSBS) of S if it satisfies the following conditions:*
(1) $f_A^p(x \boxtimes_1 y) \geq \min\{f_A^p(x), f_A^p(y)\}$,
(2) $f_A^p(x \boxtimes_2 y) \geq \min\{f_A^p(x), f_A^p(y)\}$,
(3) $f_A^p(x \boxtimes_3 y) \geq \min\{f_A^p(x), f_A^p(y)\}$,
(4) $f_A^n(x \boxtimes_1 y) \leq \max\{f_A^n(x), f_A^n(y)\}$,
(5) $f_A^n(x \boxtimes_2 y) \leq \max\{f_A^n(x), f_A^n(y)\}$,
(6) $f_A^n(x \boxtimes_3 y) \leq \max\{f_A^n(x), f_A^n(y)\}$, $\forall x, y \in S$.

Example 3.1. *Let* $S = \{x_1, x_2, x_3, x_4\}$ *be the bisemiring with the following Cayley table:*

$\boxtimes_1$	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_1	x_2
x_3	x_1	x_1	x_3	x_3
x_4	x_1	x_2	x_3	x_4

$\boxtimes_2$	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_2	x_4	x_4
x_3	x_3	x_4	x_3	x_4
x_4	x_4	x_4	x_4	x_4

$\boxtimes_3$	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_3	x_4
x_3	x_4	x_4	x_4	x_4
x_4	x_4	x_4	x_4	x_4

$$\langle f_A^p, f_A^n \rangle(x) = \begin{cases} \langle 0.70, -0.40 \rangle & \textit{if } x = x_1 \\ \langle 0.60, -0.30 \rangle & \textit{if } x = x_2 \\ \langle 0.30, -0.10 \rangle & \textit{if } x = x_3 \\ \langle 0.50, -0.20 \rangle & \textit{if } x = x_4 \end{cases}$$

Now $f_A^p(x_2 \boxtimes_1 x_3) = f_A^p(x_1) = 0.70$ *and* $\min\{f_A^p(x_2), f_A^p(x_3)\} = \min\{0.60, 0.30\} = 0.30$. *Hence,* $f_A^p(x_2 \boxtimes_1 x_3) \geq \min\{f_A^p(x_2), f_A^p(x_3)\}$. *Also,* $f_A^n(x_2 \boxtimes_1 x_3) = f_A^n(x_1) = -0.40$, $\max\{f_A^n(x_2), f_A^n(x_3)\} = \max\{-0.30, -0.10\} = -0.10$.

Hence, $f_A^n(x_2 \boxtimes_1 x_3) \leq \max\{f_A^n(x_2), f_A^n(x_3)\}$. *By routine calculations based on Definition 3.1, all the conditions are satisfied. Therefore, A is a BVSBS of S.*

Theorem 3.1. *The arbitrary intersection of a BVSBS of S is a BVSBS of S.*

Proof: Let $\{V_i \mid i \in I\}$ be the family of BVSBSs of S and $A = \bigcap_{i \in I} V_i$. Let $x, y \in S$. Then

$$f_A^p(x \boxtimes_1 y) = \inf_{i \in I}\{f_{V_i}^p(x \boxtimes_1 y)\}$$

$$\geq \inf_{i\in I}\{\min\{f^p_{V_i}(x), f^p_{V_i}(y)\}\}$$
$$= \min\left\{\inf_{i\in I}\{f^p_{V_i}(x)\}, \inf_{i\in I}\{f^p_{V_i}(y)\}\right\}$$
$$= \min\{f^p_A(x), f^p_A(y)\}.$$

Similarly, $f^p_A(x \boxtimes_2 y) \geq \min\{f^p_A(x), f^p_A(y)\}$ and $f^p_A(x \boxtimes_3 y) \geq \min\{f^p_A(x), f^p_A(y)\}$. Also,

$$\begin{aligned} f^n_A(x \boxtimes_1 y) &= \sup_{i\in I}\{f^n_{V_i}(x \boxtimes_1 y)\} \\ &\leq \sup_{i\in I}\{\max\{f^n_{V_i}(x), f^n_{V_i}(y)\}\} \\ &= \max\left\{\sup_{i\in I}\{f^n_{V_i}(x)\}, \sup_{i\in I}\{f^n_{V_i}(y)\}\right\} \\ &= \max\{f^n_A(x), f^n_A(y)\}. \end{aligned}$$

Similarly, $f^n_A(x \boxtimes_2 y) \leq \max\{f^n_A(x), f^n_A(y)\}$ and $f^n_A(x \boxtimes_3 y) \leq \max\{f^n_A(x), f^n_A(y)\}$. Hence, A is a BVSBS of S. □

Theorem 3.2. *If A and B are the two BVSBSs of S_1 and S_2, respectively, then the Cartesian product $A \times B$ is a BVSBS of $S_1 \times S_2$.*

Proof: Let A and B be the two BVSBSs of S_1 and S_2, respectively. Let $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$. Then

$$\begin{aligned} f^p_{A\times B}[(x_1, y_1) \boxtimes_1 (x_2, y_2)] &= f^p_{A\times B}(x_1 \boxtimes_1 x_2, y_1 \boxtimes_1 y_2) \\ &= \min\{f^p_A(x_1 \boxtimes_1 x_2), f^p_B(y_1 \boxtimes_1 y_2)\} \\ &\geq \min\{\min\{f^p_A(x_1), f^p_A(x_2)\}, \min\{f^p_B(y_1), f^p_B(y_2)\}\} \\ &= \min\{\min\{f^p_A(x_1), f^p_B(y_1)\}, \min\{f^p_A(x_2), f^p_B(y_2)\}\} \\ &= \min\{f^p_{A\times B}(x_1, y_1), f^p_{A\times B}(x_2, y_2)\}. \end{aligned}$$

Also,

$$f^p_{A\times B}[(x_1, y_1) \boxtimes_2 (x_2, y_2)] \geq \min\{f^p_{A\times B}(x_1, y_1), f^p_{A\times B}(x_2, y_2)\}$$

and

$$f^p_{A\times B}\,[(x_1, y_1) \boxtimes_3 (x_2, y_2)] \geq \min\{f^p_{A\times B}(x_1, y_1), f^p_{A\times B}(x_2, y_2)\}.$$

Similarly,

$$\begin{aligned} f^n_{A\times B}\,[(x_1, y_1) \boxtimes_1 (x_2, y_2)] &= f^n_{A\times B}(x_1 \boxtimes_1 x_2, y_1 \boxtimes_1 y_2) \\ &= \max\{f^n_A(x_1 \boxtimes_1 x_2), f^n_B(y_1 \boxtimes_1 y_2)\} \\ &\leq \max\{\max\{f^n_A(x_1), f^n_A(x_2)\}, \max\{f^n_B(y_1), f^n_B(y_2)\}\} \\ &= \max\{\max\{f^n_A(x_1), f^n_B(y_1)\}, \max\{f^n_A(x_2), f^n_B(y_2)\}\} \\ &= \max\{f^n_{A\times B}(x_1, y_1), f^n_{A\times B}(x_2, y_2)\}. \end{aligned}$$

Also,

$$f^n_{A\times B}\,[(x_1, y_1) \boxtimes_2 (x_2, y_2)] \leq \max\{f^n_{A\times B}(x_1, y_1), f^n_{A\times B}(x_2, y_2)\}$$

and

$$f^n_{A\times B}\,[(x_1, y_1) \boxtimes_3 (x_2, y_2)] \leq \max\{f^n_{A\times B}(x_1, y_1), f^n_{A\times B}(x_2, y_2)\}.$$

Hence, $A \times B$ is a BVSBS of $S_1 \times S_2$. □

Corollary 3.1. *If $A_1, A_2, \ldots, A_n$ are BVSBSs of bisemirings $S_1, S_2, \ldots, S_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is a BVSBS of $S_1 \times S_2 \times \cdots \times S_n$.*

Definition 3.2. *Let A be the BVFS in S, the strongest bipolar-valued relation on S, that is a bipolar-valued relation V on A is given by $f_V^p(x,y) = \min\{f_A^p(x), f_A^p(y)\}$ and $f_V^n(x,y) = \max\{f_A^n(x), f_A^n(y)\}$, $\forall x, y \in S$.*

Theorem 3.3. *Let A be the BVSBS of S and V be the strongest bipolar-valued relation on S. Then A is a BVSBS of S if and only if V is a BVSBS of $S \times S$.*

Proof: Assume that A is a BVSBS of S and V is the strongest bipolar-valued relation on S. Then for any $x = (x_1, x_2)$, $y = (y_1, y_2) \in S \times S$, we have

$$\begin{aligned} f_V^p(x \boxtimes_1 y) &= f_V^p[(x_1, x_2) \boxtimes_1 (y_1, y_2)] \\ &= f_V^p(x_1 \boxtimes_1 y_1, x_2 \boxtimes_1 y_2) \\ &= \min\{f_A^p(x_1 \boxtimes_1 y_1), f_A^p(x_2 \boxtimes_1 y_2)\} \\ &\geq \min\{\min\{f_A^p(x_1), f_A^p(y_1)\}, \min\{f_A^p(x_2), f_A^p(y_2)\}\} \\ &= \min\{\min\{f_A^p(x_1), f_A^p(x_2)\}, \min\{f_A^p(y_1), f_A^p(y_2)\}\} \\ &= \min\{f_V^p(x_1, x_2), f_V^p(y_1, y_2)\} \\ &= \min\{f_V^p(x), f_V^p(y)\}. \end{aligned}$$

Also, $f_V^p(x \boxtimes_2 y) \geq \min\{f_V^p(x), f_V^p(y)\}$ and $f_V^p(x \boxtimes_3 y) \geq \min\{f_V^p(x), f_V^p(y)\}$. Similarly,

$$f_V^n(x \boxtimes_1 y) \leq \max\{f_V^n(x), f_V^n(y)\}, \quad f_V^n(x \boxtimes_2 y) \leq \max\{f_V^n(x), f_V^n(y)\},$$

and

$$f_V^n(x \boxtimes_3 y) \leq \max\{f_V^n(x), f_V^n(y)\}.$$

Hence, V is a BVSBS of $S \times S$.

Conversely, assume that V is a BVSBS of $S \times S$. Then for any $x = (x_1, x_2)$, $y = (y_1, y_2) \in S \times S$, we have

$$\begin{aligned} \min\{f_A^p(x_1 \boxtimes_1 y_1), f_A^p(x_2 \boxtimes_1 y_2)\} &= f_V^p(x_1 \boxtimes_1 y_1, x_2 \boxtimes_1 y_2) \\ &= f_V^p[(x_1, x_2) \boxtimes_1 (y_1, y_2)] \\ &= f_V^p(x \boxtimes_1 y) \\ &\geq \min\{f_V^p(x), f_V^p(y)\} \\ &= \min\{f_V^p(x_1, x_2), f_V^p(y_1, y_2)\} \\ &= \min\{\min\{f_A^p(x_1), f_A^p(x_2)\}, \min\{f_A^p(y_1), f_A^p(y_2)\}\}. \end{aligned}$$

If $f_A^p(x_1 \boxtimes_1 y_1) \leq f_A^p(x_2 \boxtimes_1 y_2)$, then $f_A^p(x_1) \leq f_A^p(x_2)$ and $f_A^p(y_1) \leq f_A^p(y_2)$. We get $f_A^p(x_1 \boxtimes_1 y_1) \geq \min\{f_A^p(x_1), f_A^p(y_1)\}$ and $\min\{f_A^p(x_1 \boxtimes_2 y_1), f_A^p(x_2 \boxtimes_2 y_2)\} \geq \min\{\min\{f_A^p(x_1), f_A^p(x_2)\}, \min\{f_A^p(y_1), f_A^p(y_2)\}\}$.

If $f_A^p(x_1 \boxtimes_2 y_1) \leq f_A^p(x_2 \boxtimes_2 y_2)$, then $f_A^p(x_1 \boxtimes_2 y_1) \geq \min\{f_A^p(x_1), f_A^p(y_1)\}$. We get $\min\{f_A^p(x_1 \boxtimes_3 y_1), f_A^p(x_2 \boxtimes_3 y_2)\} \geq \min\{\min\{f_A^p(x_1), f_A^p(x_2)\}, \min\{f_A^p(y_1), f_A^p(y_2)\}\}$.

If $f_A^p(x_1 \boxtimes_3 y_1) \leq f_A^p(x_2 \boxtimes_3 y_2)$, then $f_A^p(x_1 \boxtimes_3 y_1) \geq \min\{f_A^p(x_1), f_A^p(y_1)\}$. Similarly, $\max\{f_A^n(x_1 \boxtimes_1 y_1), f_A^n(x_2 \boxtimes_1 y_2)\} \leq \max\{\max\{f_A^n(x_1), f_A^n(x_2)\}, \max\{f_A^n(y_1), f_A^n(y_2)\}\}$.

If $f_A^n(x_1 \boxtimes_1 y_1) \geq f_A^n(x_2 \boxtimes_1 y_2)$, then $f_A^n(x_1) \geq f_A^n(x_2)$ and $f_A^n(y_1) \geq f_A^n(y_2)$. We get $f_A^n(x_1 \boxtimes_1 y_1) \leq \max\{f_A^n(x_1), f_A^n(y_1)\}$, so $\max\{f_A^n(x_1 \boxtimes_2 y_1), f_A^n(x_2 \boxtimes_2 y_2)\} \leq \max\{\max\{f_A^n(x_1), f_A^n(x_2)\}, \max\{f_A^n(y_1), f_A^n(y_2)\}\}$.

If $f_A^n(x_1 \boxtimes_2 y_1) \geq f_A^n(x_2 \boxtimes_2 y_2)$, then $f_A^n(x_1 \boxtimes_2 y_1) \leq \max\{f_A^n(x_1), f_A^n(y_1)\}$. We get $\max\{f_A^n(x_1 \boxtimes_3 y_1), f_A^n(x_2 \boxtimes_3 y_2)\} \leq \max\{\max\{f_A^n(x_1), f_A^n(x_2)\}, \max\{f_A^n(y_1), f_A^n(y_2)\}\}$.

If $f_A^n(x_1 \boxtimes_3 y_1) \geq f_A^n(x_2 \boxtimes_3 y_2)$, then $f_A^n(x_1 \boxtimes_3 y_1) \leq \max\{f_A^n(x_1), f_A^n(y_1)\}$.

Hence, A is a BVSBS of S. □

Theorem 3.4. *A BVFS $\tilde{f} = \langle f_A^p, f_A^n \rangle$ is an BVSBS of S if and only if all non-empty level set $\tilde{f}^{(t,s)}$ is an SBS of S for $t \in [0, 1]$ and $s \in [-1, 0]$.*

Proof: Assume that $\tilde{f}$ is a BVSBS of S. For each $t \in [0,1]$ and $s \in [-1,0]$ and $a_1, a_2 \in \tilde{f}^{(t,s)}$, we have $f_A^p(a_1) \geq t$, $f_A^p(a_2) \geq t$ and $f_A^n(a_1) \leq s$, $f_A^n(a_2) \leq s$. Now, $f_A^p(a_1 \boxtimes_1 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\} \geq t$, $f_A^p(a_1 \boxtimes_2 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\} \geq t$, and $f_A^p(a_1 \boxtimes_3 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\} \geq t$. Similarly,

$$f_A^n(a_1 \boxtimes_1 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\} \leq s, \ f_A^n(a_1 \boxtimes_2 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\} \leq s,$$

and

$$f_A^n(a_1 \boxtimes_3 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\} \leq s.$$

This implies that $a_1 \boxtimes_1 a_2 \in \tilde{f}^{(t,s)}$, $a_1 \boxtimes_2 a_2 \in \tilde{f}^{(t,s)}$, and $a_1 \boxtimes_3 a_2 \in \tilde{f}^{(t,s)}$. Therefore, $\tilde{f}^{(t,s)}$ is an SBS of S for each $t \in [0,1]$ and $s \in [-1,0]$.

Conversely, assume that $\tilde{f}^{(t,s)}$ is an SBS of S for each $t \in [0,1]$ and $s \in [-1,0]$. Suppose if there exist $a_1, a_2 \in S$ such that $f_A^p(a_1 \boxtimes_1 a_2) < \min\{f_A^p(a_1), f_A^p(a_2)\}$. Select $t \in [0,1]$ such that $f_A^p(a_1 \boxtimes_1 a_2) < t \leq \min\{f_A^p(a_1), f_A^p(a_2)\}$, $f_A^p(a_1 \boxtimes_2 a_2) < t \leq \min\{f_A^p(a_1), f_A^p(a_2)\}$, and $f_A^p(a_1 \boxtimes_3 a_2) < t \leq \min\{f_A^p(a_1), f_A^p(a_2)\}$. Then $a_1, a_2 \in \tilde{f}^{(t,s)}$, but $a_1 \boxtimes_1 a_2 \notin \tilde{f}^{(t,s)}$, $a_1 \boxtimes_2 a_2 \notin \tilde{f}^{(t,s)}$, and $a_1 \boxtimes_3 a_2 \notin \tilde{f}^{(t,s)}$. This contradicts to that $\tilde{f}^{(t,s)}$ is an SBS of S. Hence, $f_A^p(a_1 \boxtimes_1 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\}$, $f_A^p(a_1 \boxtimes_2 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\}$, and $f_A^p(a_1 \boxtimes_3 a_2) \geq \min\{f_A^p(a_1), f_A^p(a_2)\}$. Similarly,

$$f_A^n(a_1 \boxtimes_1 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\}, \ f_A^n(a_1 \boxtimes_2 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\},$$

and

$$f_A^n(a_1 \boxtimes_3 a_2) \leq \max\{f_A^n(a_1), f_A^n(a_2)\}.$$

Hence, $\tilde{f} = \langle f_A^p, f_A^n \rangle$ is a BVSBS of S. □

Theorem 3.5. *If A is a BVSBS of S, then $H = \{x \mid x \in S \mid f_A^p(x) = 1$ and $f_A^n(x) = -1\}$ is either empty or is an SBS of S.*

Proof: Assume that H is non-empty. If $x, y \in H$, then $f_A^p(x) = 1$, $f_A^p(y) = 1$ and $f_A^n(x) = -1$, $f_A^n(y) = -1$. Now, $f_A^p(x \boxtimes_1 y) \geq \min\{f_A^p(x), f_A^p(y)\} = \min\{1,1\} = 1$. Therefore, $f_A^p(x \boxtimes_1 y) = 1$. Similarly, $f_A^p(x \boxtimes_2 y) = 1$ and $f_A^p(x \boxtimes_3 y) = 1$. Now, $f_A^n(x \boxtimes_1 y) \leq \max\{f_A^n(x), f_A^n(y)\} = \max\{-1,-1\} = -1$. Therefore, $f_A^n(x \boxtimes_1 y) = -1$. Similarly, $f_A^n(x \boxtimes_2 y) = -1$ and $f_A^n(x \boxtimes_3 y) = -1$. Thus, $x \boxtimes_1 y \in H$, $x \boxtimes_2 y \in H$, and $x \boxtimes_3 y \in H$. Hence, H is an SBS of S. □

Definition 3.3. *Let A be any BVSBS of S, $a \in S$ and a fixed real number $p(a) \in [0,1]$. Then the pseudo bipolar-valued coset $(aA)^p$ is defined by $((af_A^p)^p)(x) = p(a)f_A^p(x)$ and $((af_A^n)^p)(x) = p(a)f_A^n(x)$ for every $x \in S$.*

Theorem 3.6. *If A is a BVSBS of S, then the pseudo bipolar-valued coset $(aA)^p$ is a BVSBS of S for every $a \in S$.*

Proof: Let A be any BVSBS of S and for every $x, y \in S$. Then $((af_A^p)^p)(x \boxtimes_1 y) = p(a)f_A^p(x \boxtimes_1 y) \geq p(a)\min\{f_A^p(x), f_A^p(y)\} = \min\{p(a)f_A^p(x), p(a)f_A^p(y)\} = \min\{((af_A^p)^p)(x), ((af_A^p)^p)(y)\}$. Hence, $((af_A^p)^p)(x \boxtimes_1 y) \geq \min\{((af_A^p)^p)(x), ((af_A^p)^p)(y)\}$. Similarly,

$$((af_A^p)^p)(x \boxtimes_2 y) \geq \min\{((af_A^p)^p)(x), ((af_A^p)^p)(y)\}$$

and

$$((af_A^p)^p)(x \boxtimes_3 y) \geq \min\{((af_A^p)^p)(x), ((af_A^p)^p)(y)\}.$$

Now, $((af_A^n)^p)(x \boxtimes_1 y) = p(a)f_A^n(x \boxtimes_1 y) \leq p(a)\max\{f_A^n(x), f_A^n(y)\} = \max\{p(a)f_A^n(x), p(a)f_A^n(y)\} = \max\{((af_A^n)^p)(x), ((af_A^n)^p)(y)\}$. Hence, $((af_A^n)^p)(x \boxtimes_1 y) \leq \max\{((af_A^n)^p)(x), ((af_A^n)^p)(y)\}$. Similarly,

$$((af_A^n)^p)(x \boxtimes_2 y) \leq \max\{((af_A^n)^p)(x), ((af_A^n)^p)(y)\}$$

and

$$((af_A^n)^p)(x \boxtimes_3 y) \leq \max\{((af_A^n)^p)(x), ((af_A^n)^p)(y)\}.$$

Hence, $(aA)^p$ is a BVSBS of S. □

Definition 3.4. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. Let φ: $S_1 \to S_2$ be any function, A be any BVSBS in S_1, and V be any BVSBS in $\varphi(S_1) = S_2$. If $f_A = \langle f_A^p, f_A^n\rangle$ is a BVFS in S_1, then f_V is a BVFS in S_2, defined by $f_V^p(y) = \sup_{x\in\varphi^{-1}y} f_A^p(x)$ and $f_V^n(y) = \inf_{x\in\varphi^{-1}y} f_A^n(x)$ for all $x \in S_1$ and $y \in S_2$ is called the image of f_A under φ. Similarly, if $f_V = \langle f_V^p, f_V^n\rangle$ is a BVFS in S_2, then the BVFS $f_A = \varphi \circ f_V$ in S_1 [i.e., the BVFS defined by $f_A(x) = f_V(\varphi(x))$] is called the preimage of f_V under φ.*

Theorem 3.7. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic image of BVSBS of S_1 is a BVSBS of S_2.*

Proof: Let φ: $S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where A is any BVSBS of S_1. Let $\varphi(x), \varphi(y) \in S_2$. Let $x \in \varphi^{-1}(\varphi(x))$ and $y \in \varphi^{-1}(\varphi(y))$ be such that $f_A^p(x) = \sup_{z\in\varphi^{-1}(\varphi(x))} f_A^p(z)$ and $f_A^p(y) = \sup_{z\in\varphi^{-1}(\varphi(y))} f_A^p(z)$. Now,

$$\begin{aligned} f_V^p(\varphi(x) \odot_1 \varphi(y)) &= \sup_{z'\in\varphi^{-1}(\varphi(x)\odot_1\varphi(y))} f_A^p(z') \\ &= \sup_{z'\in\varphi^{-1}(\varphi(x\oplus_1 y))} f_A^p(z') \\ &= f_A^p(x \oplus_1 y) \\ &\geq \min\{f_A^p(x), f_A^p(y)\} \\ &= \min\{f_V^p\varphi(x), f_V^p\varphi(y)\}. \end{aligned}$$

Thus, $f_V^p(\varphi(x) \odot_1 \varphi(y)) \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y)\}$. Similarly,

$$f_V^p(\varphi(x) \odot_2 \varphi(y)) \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y)\}$$

and

$$f_V^p(\varphi(x) \odot_3 \varphi(y)) \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y)\}.$$

Let $\varphi(x), \varphi(y) \in S_2$. Let $x \in \varphi^{-1}(\varphi(x))$ and $y \in \varphi^{-1}(\varphi(y))$ be such that $f_A^n(x) = \inf_{z\in\varphi^{-1}(\varphi(x))} f_A^n(z)$ and $f_A^n(y) = \inf_{z\in\varphi^{-1}(\varphi(y))} f_A^n(z)$. Now,

$$\begin{aligned} f_V^n(\varphi(x) \odot_1 \varphi(y)) &= \inf_{z'\in\varphi^{-1}(\varphi(x)\odot_1\varphi(y))} f_A^n(z') \\ &= \inf_{z'\in\varphi^{-1}(\varphi(x\oplus_1 y))} f_A^n(z') \\ &= f_A^n(x \oplus_1 y) \\ &\leq \max\{f_A^n(x), f_A^n(y)\} \\ &= \max\{f_V^n\varphi(x), f_V^n\varphi(y)\}. \end{aligned}$$

Thus, $f_V^n(\varphi(x) \odot_1 \varphi(y)) \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y)\}$. Similarly,

$$f_V^n(\varphi(x) \odot_2 \varphi(y)) \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y)\}$$

and

$$f_V^n(\varphi(x) \odot_3 \varphi(y)) \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y)\}.$$

Hence, V is a BVSBS of S_2. □

Theorem 3.8. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic preimage of BVSBS of S_2 is a BVSBS of S_1.*

Proof: Let φ: $S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where V is any BVSBS of S_2. Let $x, y \in S_1$. Then $f_A^p(x \oplus_1 y) = f_V^p(\varphi(x \oplus_1 y)) = f_V^p(\varphi(x) \odot_1 \varphi(y)) \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y)\} = \min\{f_A^p(x), f_A^p(y)\}$. Thus, $f_A^p(x \oplus_1 y) \geq \min\{f_A^p(x), f_A^p(y)\}$. Similarly,

$$f_A^p(x \oplus_2 y) \geq \min\{f_A^p(x), f_A^p(y)\}$$

and

$$f_A^p(x \oplus_3 y) \geq \min\{f_A^p(x), f_A^p(y)\}.$$

Now, $f_A^n(x \oplus_1 y) = f_V^n(\varphi(x \oplus_1 y)) = f_V^n(\varphi(x) \odot_1 \varphi(y)) \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y)\} = \max\{f_A^n(x), f_A^n(y)\}$. Thus, $f_A^n(x \oplus_1 y) \leq \max\{f_A^n(x), f_A^n(y)\}$. Similarly,

$$f_A^n(x \oplus_2 y) \leq \max\{f_A^n(x), f_A^n(y)\}$$

and

$$f_A^n(x \oplus_3 y) \leq \max\{f_A^n(x), f_A^n(y)\}.$$

Hence, A is a BVSBS of S_1. □

Theorem 3.9. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If φ: $S_1 \to S_2$ is a homomorphism, then $\varphi(A_{(t,s)})$ is a level SBS of BVSBS V of S_2.*

Proof: Let φ: $S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where A is a BVSBS of S_1. By Theorem 3.7, we have V is a BVSBS of S_2. Let $A_{(t,s)}$ be any level SBS of A. Suppose that $x, y \in A_{(t,s)}$. Then $\varphi(x \oplus_1 y), \varphi(x \oplus_2 y)$ and $\varphi(x \oplus_3 y) \in A_{(t,s)}$. Now, $f_V^p(\varphi(x)) = f_A^p(x) \geq t$ and $f_V^p(\varphi(y)) = f_A^p(y) \geq t$. Then $f_V^p(\varphi(x) \odot_1 \varphi(y)) \geq f_A^p(x \oplus_1 y) \geq t$, $f_V^p(\varphi(x) \odot_2 \varphi(y)) \geq f_A^p(x \oplus_2 y) \geq t$, and $f_V^p(\varphi(x) \odot_3 \varphi(y)) \geq f_A^p(x \oplus_3 y) \geq t$ for all $\varphi(x), \varphi(y) \in S_2$. Now, $f_V^n(\varphi(x)) = f_A^n(x) \leq s$ and $f_V^n(\varphi(y)) = f_A^n(y) \leq s$. Then $f_V^n(\varphi(x) \odot_1 \varphi(y)) \leq f_A^n(x \oplus_1 y) \leq s$, $f_V^n(\varphi(x) \odot_2 \varphi(y)) \leq f_A^n(x \oplus_2 y) \leq s$, and $f_V^n(\varphi(x) \odot_3 \varphi(y)) \leq f_A^n(x \oplus_3 y) \leq s$ for all $\varphi(x), \varphi(y) \in S_2$. Hence, $\varphi\left(A_{(t,s)}\right)$ is a level SBS of BVSBS V of S_2. □

Theorem 3.10. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. If φ: $S_1 \to S_2$ is any homomorphism, then $A_{(t,s)}$ is a level SBS of BVSBS A of S_1.*

Proof: Let φ: $S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where V is a BVSBS of S_2. By Theorem 3.8, we have A is a BVSBS of S_1. Let $\varphi(A_{(t,s)})$ be a level SBS of V. Suppose that $\varphi(x), \varphi(y) \in \varphi(A_{(t,s)})$. Then $\varphi(x \oplus_1 y), \varphi(x \oplus_2 y), \varphi(x \oplus_3 y) \in \varphi(A_{(t,s)})$. Now, $f_A^p(x) = f_V^p(\varphi(x)) \geq t$ and $f_A^p(y) = f_V^p(\varphi(y)) \geq t$. Then $f_A^p(x \oplus_1 y) \geq t$, $f_A^p(x \oplus_2 y) \geq t$, and $f_A^p(x \oplus_3 y) \geq t$ for all $x, y \in S_1$. Now, $f_A^n(x) = f_V^n(\varphi(x)) \leq s$ and $f_A^n(y) = f_V^n(\varphi(y)) \leq s$. Then $f_A^n(x \oplus_1 y) = f_V^n(\varphi(x) \odot_1 \varphi(y)) \leq s$, $f_A^n(x \oplus_2 y) = f_V^n(\varphi(x) \odot_2 \varphi(y)) \leq s$, and $f_A^n(x \oplus_3 y) = f_V^n(\varphi(x) \odot_3 \varphi(y)) \leq s$ for all $x, y \in S_1$. Hence, $A_{(t,s)}$ is a level SBS of BVSBS A of S_1. □

4. $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-Bipolar-Valued Subbisemirings.

In what follows that, let $(\alpha^p, \beta^p) \in [0, 1]$ and $(\alpha^n, \beta^n) \in [-1, 0]$ be such that $0 \leq \alpha^p < \beta^p \leq 1$ and $-1 \leq \beta^n < \alpha^n \leq 0$, both $(\alpha, \beta) \in [0, 1]$ are arbitrary fixed.

Definition 4.1. *Let S be the SBS. The BVFS A in S is called an (α, β)-bipolar-valued subbisemiring ((α, β)-BVSBS) of S if it satisfies the following conditions:*

(1) $\max\{f_A^p(x \boxtimes_1 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$,

(2) $\max\{f_A^p(x \boxtimes_2 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$,

(3) $\max\{f_A^p(x \boxtimes_3 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$,

(4) $\min\{f_A^n(x \boxtimes_1 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}$,

(5) $\min\{f_A^n(x \boxtimes_2 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}$,

(6) $\min\{f_A^n(x \boxtimes_3 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}$, $\forall x, y \in S$.

Example 4.1. *By Example 3.1, we have*

$$\langle f^p, f^n\rangle(x) = \begin{cases} \langle 0.75, -0.45\rangle & \text{if } x = x_1 \\ \langle 0.65, -0.35\rangle & \text{if } x = x_2 \\ \langle 0.35, -0.15\rangle & \text{if } x = x_3 \\ \langle 0.55, -0.25\rangle & \text{if } x = x_4 \end{cases}$$

Then A is a $(0.60, 0.70)$-BVSBS of S.

Theorem 4.1. *The arbitrary intersection of an (α, β)-BVSBSs of S is an (α, β)-BVSBS of S.*

Proof: Let $\{V_i \mid i \in I\}$ be a family of (α, β)-BVSBSs of S and $A = \bigcap_{i\in I} V_i$. Let $x, y \in S$. Then

$$\begin{aligned} \max\{f_A^p(x \boxtimes_1 y), \alpha^p\} &= \inf_{i\in I}\{\max\{f_{V_i}^p(x \boxtimes_1 y), \alpha^p\}\} \\ &\geq \inf_{i\in I}\{\min\{f_{V_i}^p(x), f_{V_i}^p(y), \beta^p\}\} \\ &= \min\left\{\inf_{i\in I}\{f_{V_i}^p(x)\}, \inf_{i\in I}\{f_{V_i}^p(y), \beta^p\}\right\} \\ &= \min\{f_A^p(x), f_A^p(y), \beta^p\}. \end{aligned}$$

Similarly,

$$\max\{f_A^p(x \boxtimes_2 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$$

and

$$\max\{f_A^p(x \boxtimes_3 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}.$$

Now,

$$\begin{aligned} \min\{f_A^n(x \boxtimes_1 y), \alpha^n\} &= \sup_{i\in I}\{\min\{f_{V_i}^n(x \boxtimes_1 y), \alpha^n\}\} \\ &\leq \sup_{i\in I}\{\max\{f_{V_i}^n(x), f_{V_i}^n(y), \beta^n\}\} \\ &= \max\left\{\sup_{i\in I}\{f_{V_i}^n(x)\}, \sup_{i\in I}\{f_{V_i}^n(y), \beta^n\}\right\} \\ &= \max\{f_A^n(x), f_A^n(y), \beta^n\}. \end{aligned}$$

Similarly,

$$\min\{f_A^n(x \boxtimes_2 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}$$

and

$$\min\{f_A^n(x \boxtimes_3 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}.$$

Hence, A is an (α, β)-BVSBS of S. □

Theorem 4.2. *If A and B are any two (α, β)-BVSBSs of bisemirings S_1 and S_2, respectively, then $A \times B$ is an (α, β)-BVSBS of $S_1 \times S_2$.*

Proof: Let A and B be two (α, β)-BVSBSs of S_1 and S_2, respectively. Let $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$. Then

$$\begin{aligned}
&\max\{f^p_{A\times B}[(x_1,y_1)\boxtimes_1(x_2,y_2)],\alpha^p\}\\
&=\max\{f^p_{A\times B}(x_1\boxtimes_1 x_2, y_1\boxtimes_1 y_2),\alpha^p\}\\
&=\min\{\max\{f^p_A(x_1\boxtimes_1 x_2),\alpha^p\},\max\{f^p_B(y_1\boxtimes_1 y_2),\alpha^p\}\}\\
&\geq\min\{\min\{f^p_A(x_1),f^p_A(x_2),\beta^p\},\min\{f^p_B(y_1),f^p_B(y_2),\beta^p\}\}\\
&=\min\{\{\min\{f^p_A(x_1),f^p_B(y_1)\},\min\{f^p_A(x_2),f^p_B(y_2)\}\},\beta^p\}\\
&=\min\{f^p_{A\times B}(x_1,y_1),f^p_{A\times B}(x_2,y_2),\beta^p\}.
\end{aligned}$$

Also,

$$\max\{f^p_{A\times B}[(x_1,y_1)\boxtimes_2(x_2,y_2)],\alpha^p\}\geq\min\{f^p_{A\times B}(x_1,y_1),f^p_{A\times B}(x_2,y_2),\beta^p\}$$

and

$$\max\{f^p_{A\times B}[(x_1,y_1)\boxtimes_3(x_2,y_2)],\alpha^p\}\geq\min\{f^p_{A\times B}(x_1,y_1),f^p_{A\times B}(x_2,y_2),\beta^p\}.$$

Similarly,

$$\begin{aligned}
&\min\{f^n_{A\times B}[(x_1,y_1)\boxtimes_1(x_2,y_2)],\alpha^n\}\\
&=\min\{f^n_{A\times B}(x_1\boxtimes_1 x_2, y_1\boxtimes_1 y_2),\alpha^n\}\\
&=\max\{\min\{f^n_A(x_1\boxtimes_1 x_2),\alpha^n\},\min\{f^n_B(y_1\boxtimes_1 y_2),\alpha^n\}\}\\
&\leq\max\{\max\{f^n_A(x_1),f^n_A(x_2),\beta^n\},\max\{f^n_B(y_1),f^n_B(y_2),\beta^n\}\}\\
&=\max\{\{\max\{f^n_A(x_1),f^n_B(y_1)\},\max\{f^n_A(x_2),f^n_B(y_2)\}\},\beta^n\}\\
&=\max\{f^n_{A\times B}(x_1,y_1),f^n_{A\times B}(x_2,y_2),\beta^n\}.
\end{aligned}$$

Also,

$$\min\{f^n_{A\times B}[(x_1,y_1)\boxtimes_2(x_2,y_2)],\alpha^n\}\leq\max\{f^n_{A\times B}(x_1,y_1),f^n_{A\times B}(x_2,y_2),\beta^n\}$$

and

$$\min\{f^n_{A\times B}[(x_1,y_1)\boxtimes_3(x_2,y_2)],\alpha^n\}\leq\max\{f^n_{A\times B}(x_1,y_1),f^n_{A\times B}(x_2,y_2),\beta^n\}.$$

Hence, $A\times B$ is an (α,β)-BVSBS of $S_1\times S_2$. □

Corollary 4.1. *If $A_1, A_2, \dots, A_n$ are (α,β)-BVSBSs of $S_1, S_2, \dots, S_n$, respectively, then $A_1\times A_2\times\cdots\times A_n$ is an (α,β)-BVSBS of $S_1\times S_2\times\cdots\times S_n$.*

Definition 4.2. *Let A be a BVFS in S, the strongest (α,β)-bipolar-valued relation on S, that is an (α,β)-bipolar-valued relation on A is V given by $\max\{f^p_V(x,y),\alpha^p\}=\min\{f^p_A(x),f^p_A(y),\beta^p\}$ and $\min\{f^n_V(x,y),\alpha^n\}=\max\{f^n_A(x),f^n_A(y),\beta^n\}$ for all $x,y\in S$.*

Theorem 4.3. *Let A be an (α,β)-BVSBS of S and V be the strongest (α,β)-bipolar-valued relation on S. Then A is an (α,β)-BVSBS of S if and only if V is an (α,β)-BVSBS of $S\times S$.*

Proof: The proof is similar to Theorem 3.3. □

Theorem 4.4. *If $f_{\tilde{\alpha}}$ is an (α,β)-BVSBS of S, then the nonempty sets f^p_α and f^n_α are SBSs of S, where $f^p_\alpha=\{p\in S\mid f^p(p)>\alpha^p\}$ and $f^n_\alpha=\{p\in S\mid f^n(p)<\alpha^n\}$.*

Proof: Suppose that $f_{\tilde{\alpha}}$ is an (α,β)-BVSBS of S. Let f^p_α be an (α^p,β^p)-BVSBS of S. Let $p,q\in S$ be such that $p,q\in f^p_\alpha$. Then $f^p(p)>\alpha^p$ and $f^p(q)>\alpha^p$. Now, $\max\{f^p(p\boxtimes_1 q),\alpha^p\}\geq\min\{f^p(p),f^p(q),\beta^p\}>\min\{\alpha^p,\alpha^p,\beta^p\}=\alpha^p$. Hence, $f^p(p\boxtimes_1 q)>\alpha^p$. It shows that $p\boxtimes_1 q\in f^p_\alpha$. Similarly, $p\boxtimes_2 q\in f^p_\alpha$ and $p\boxtimes_3 q\in f^p_\alpha$. Therefore, f^p_α is an SBS of S. Let f^n_α be an (α^n,β^n)-BVSBS of S. Let $p,q\in S$ be such that $p,q\in f^n_\alpha$. Then $f^n(p)<\alpha^n$ and $f^n(q)<\alpha^n$. Now, $\min\{f^n(p\boxtimes_1 q),\alpha^n\}\leq\max\{f^n(p),f^n(q),\beta^n\}<\max\{\alpha^n,\alpha^n,\beta^n\}=\alpha^n$. Hence, $f^n(p\boxtimes_1 q)<\alpha^n$. It shows that $p\boxtimes_1 q\in f^n_\alpha$. Similarly, $p\boxtimes_2 q\in f^n_\alpha$ and $p\boxtimes_3 q\in f^n_\alpha$. Therefore, f^n_α is an SBS of S. □

Theorem 4.5. *A non-empty subset A of S is an SBS of S if and only if the BVFS $\tilde{f} = \langle f_A^p, f_A^n \rangle$ of S, and then is an (α, β)-BVSBS of S, where*

$$f_A^p(p) = \begin{cases} \geq \beta^p & \text{for all } p \in A \\ \alpha^p & \text{otherwise} \end{cases}, \quad f_A^n(p) = \begin{cases} \leq \beta^n & \text{for all } p \in A \\ \alpha^n & \text{otherwise} \end{cases}$$

Proof: Suppose that $\tilde{f} = \langle f_A^p, f_A^n \rangle$ is an (α, β)-BVSBS of S. Let $p, q \in A$. Then $f_A^p(p) \geq \beta^p$, $f_A^p(q) \geq \beta^p$ and $f_A^n(p) \leq \beta^n$, $f_A(q) \leq \beta^n$. Now, $\max\{f_A^p(p \boxtimes_1 q), \alpha^p\} \geq \min\{f_A^p(p), f_A^p(q), \beta^p\} \geq \min\{\beta^p, \beta^p, \beta^p\} = \beta^p$ and $\min\{f_A^n(p \boxtimes_1 q), \alpha^n\} \leq \max\{f_A^n(p), f_A^n(q), \beta^n\} \leq \max\{\beta^n, \beta^n, \beta^n\} = \beta^n$. It follows that $p \boxtimes_1 q \in A$. Similarly, $p \boxtimes_2 q \in A$ and $p \boxtimes_3 q \in A$. If we choose $p, q \notin A$, then $p \boxtimes_1 q \in A$, $p \boxtimes_2 q \in A$, and $p \boxtimes_3 q \in A$. Therefore, A is an SBS of S.

Conversely, suppose that A is an SBS of S. Let $p, q \in A$. Then $p \boxtimes_1 q \in A$. Hence, $f_A^p(p \boxtimes_1 q) \geq \beta^p$ and $f_A^n(p \boxtimes_1 q) \leq \beta^n$. Therefore, $\max\{f_A^p(p \boxtimes_1 q), \alpha^p\} \geq \beta^p = \min\{f_A^p(p), f_A^p(q), \beta^p\}$ and $\min\{f_A^n(p \boxtimes_1 q), \alpha^n\} \leq \beta^n = \max\{f_A^n(p), f_A^n(q), \beta^n\}$. If $p \notin A$ or $q \notin A$, then $\min\{f_A^p(p), f_A^p(q), \beta^p\} = \alpha^p$ and $\max\{f_A^n(p), f_A^n(q), \beta^n\} = \alpha^n$. That is $\max\{f_A^p(p \boxtimes_1 q), \alpha^p\} \geq \min\{f_A^p(p), f_A^p(q), \beta^p\}$ and $\min\{f_A^n(p \boxtimes_1 q), \alpha^n\} \leq \max\{f_A^n(p), f_A^n(q), \beta^n\}$. Similarly, other two operations $\boxtimes_2$ and $\boxtimes_3$ are true. Therefore, $\tilde{f}$ is an (α, β)-BVSBS of S. □

Theorem 4.6. *A BVFS $\tilde{f} = \langle f_A^p, f_A^n \rangle$ is an (α, β)-BVSBS of S if and only if each non-empty level subset $\tilde{f}^{(t,s)}$ is an SBS of S for all $t \in (\alpha^p, \beta^p]$ and $s \in (\alpha^n, \beta^n]$.*

Proof: Suppose that $\tilde{f}$ is an (α, β)-BVSBS of S. For each $t \in (\alpha^p, \beta^p]$ and $s \in (\alpha^n, \beta^n]$ and $p_1, p_2 \in \tilde{f}^{(t,s)}$, we have $f_A^p(p_1) \geq t$, $f_A^p(p_2) \geq t$ and $f_A^n(p_1) \leq s$, $f_A^n(p_2) \leq s$. Now, $\max\{f_A^p(p_1 \boxtimes_1 p_2), \alpha^p\} \geq \min\{f_A^p(p_1), f_A^p(p_2, \beta^p)\} \geq t$ and $\max\{f_A^p(p_1 \boxtimes_2 p_2), \alpha^p\} \geq t$ and $\max\{f_A^p(p_1 \boxtimes_3 p_2), \alpha^p\} \geq t$. Similarly,

$$\min\{f_A^n(p_1 \boxtimes_1 p_2), \alpha^n\} \leq \max\{f_A^n(p_1), f_A^n(p_2), \beta^n)\} \leq s, \ \min\{f_A^n(p_1 \boxtimes_2 p_2), \alpha^n\} \leq s,$$

and

$$\min\{f_A^n(p_1 \boxtimes_3 p_2), \alpha^n\} \leq s.$$

This implies that $p_1 \boxtimes_1 p_2 \in \tilde{f}^{(t,s)}$, $p_1 \boxtimes_2 p_2 \in \tilde{f}^{(t,s)}$, and $p_1 \boxtimes_3 p_2 \in \tilde{f}^{(t,s)}$. Therefore, $\tilde{f}^{(t,s)}$ is an SBS of S for each $t \in (\alpha^p, \beta^p]$ and $s \in (\alpha^n, \beta^n]$.

Conversely, suppose that $\tilde{f}^{(t,s)}$ is any SBS of S for each $t \in (\alpha^p, \beta^p]$ and $s \in (\alpha^n, \beta^n]$. Suppose if there exist $p_1, p_2 \in S$ such that $\max\{f_A^p(p_1 \boxtimes_1 p_2), \alpha^p\} < \min\{f_A^p(p_1), f_A^p(p_2), \beta^p\}$. Select $t \in [0, 1]$ and $s \in [-1, 0]$ such that $\max\{f_A^p(p_1 \boxtimes_1 p_2), \alpha^p\} < t \leq \min\{f_A^p(p_1), f_A^p(p_2), \beta^p\}$ and $\min\{f_A^n(p_1 \boxtimes_1 p_2), \alpha^n\} > s \geq \max\{f_A^n(p_1), f_A^n(p_2), \beta^n\}$. Then $p_1, p_2 \in \tilde{f}^{(t,s)}$, but $p_1 \boxtimes_1 p_2 \notin \tilde{f}^{(t,s)}$. This contradicts to that $\tilde{f}^{(t,s)}$ is an SBS of S. Hence, $\max\{f_A^p(p_1 \boxtimes_1 p_2), \alpha^p\} \geq \min\{f_A^p(p_1), f_A^p(p_2), \beta^p\}$ and $\min\{f_A^n(p_1 \boxtimes_1 p_2), \alpha^n\} \leq \max\{f_A^n(p_1), f_A^n(p_2), \beta^n\}$. Similar proof for other two operations. Hence, $\tilde{f}$ is an (α, β)-BVSBS of S. □

Corollary 4.2. *Every BVSBS is an (α, β)-BVSBS of S by taking $\alpha^p = 0$, $\beta^p = 1$ and $\alpha^n = 0$, $\beta^n = -1$. However, converse is not true by the following example.*

Example 4.2. *For Example 3.1, we define the BVFS $\tilde{f}$ as follows:*

$$\langle f_A^p, f_A^n \rangle(x) = \begin{cases} \langle 0.80, -0.60 \rangle & \text{if } x = a_1 \\ \langle 0.70, -0.50 \rangle & \text{if } x = a_2 \\ \langle 0.50, -0.30 \rangle & \text{if } x = a_3 \\ \langle 0.30, -0.20 \rangle & \text{if } x = a_4 \end{cases}$$

Then $\tilde{f}$ is a $(0.65, 0.75)$-BVSBS of S, but not a BVSBS. Since $f_A^p(a_3 \boxtimes_3 a_3) = f_A^p(a_4) = 0.30 \ngeq \min\{f_A^p(a_3), f_A^p(a_3)\} = 0.50$ and $f_A^n(a_3 \boxtimes_3 a_3) = f_A^n(a_4) = -0.20 \nleq \max\{f_A^n(a_3), f_A^n(a_3)\} = -0.30$.

Theorem 4.7. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic image of (α, β)-BVSBS of S_1 is an (α, β)-BVSBS of S_2.*

Proof: Let $\varphi\colon S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where A is any (α, β)-BVSBS of S_1. Let $\varphi(x), \varphi(y) \in S_2$. Let $x \in \varphi^{-1}(\varphi(x))$ and $y \in \varphi^{-1}(\varphi(y))$ be such that $f_A^p(x) = \sup_{z \in \varphi^{-1}(\varphi(x))} f_A^p(z)$ and $f_A^p(y) = \sup_{z \in \varphi^{-1}(\varphi(y))} f_A^p(z)$. Now,

$$\begin{aligned}
\max\{f_V^p(\varphi(x) \odot_1 \varphi(y)), \alpha^p\} &= \max\left\{\sup_{z' \in \varphi^{-1}(\varphi(x) \odot_1 \varphi(y))} f_A^p(z'), \alpha^p\right\} \\
&= \max\left\{\sup_{z' \in \varphi^{-1}(\varphi(x \oplus_1 y))} f_A^p(z'), \alpha^p\right\} \\
&= \max\{f_A^p(x \oplus_1 y), \alpha^p\} \\
&\geq \min\{f_A^p(x), f_A^p(y), \beta^p\} \\
&= \min\{f_V^p\varphi(x), f_V^p\varphi(y), \beta^p\}.
\end{aligned}$$

Thus, $\max\{f_V^p(\varphi(x) \odot_1 \varphi(y)), \alpha^p\} \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y), \beta^p\}$. Similarly,

$$\max\{f_V^p(\varphi(x) \odot_2 \varphi(y)), \alpha^p\} \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y), \beta^p\}$$

and

$$\max\{f_V^p(\varphi(x) \odot_3 \varphi(y)), \alpha^p\} \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y), \beta^p\}.$$

Let $x \in \varphi^{-1}(\varphi(x))$ and $y \in \varphi^{-1}(\varphi(y))$ be such that $f_A^n(x) = \inf_{z \in \varphi^{-1}(\varphi(x))} f_A^n(z)$ and $f_A^n(y) = \inf_{z \in \varphi^{-1}(\varphi(y))} f_A^n(z)$. Now,

$$\begin{aligned}
\min\{f_V^n(\varphi(x) \odot_1 \varphi(y)), \alpha^n\} &= \min\left\{\inf_{z' \in \varphi^{-1}(\varphi(x) \odot_1 \varphi(y))} f_A^n(z'), \alpha^n\right\} \\
&= \min\left\{\inf_{z' \in \varphi^{-1}(\varphi(x \oplus_1 y))} f_A^n(z'), \alpha^n\right\} \\
&= \min\{f_A^n(x \oplus_1 y), \alpha^n\} \\
&\leq \max\{f_A^n(x), f_A^n(y), \beta^n\} \\
&= \max\{f_V^n\varphi(x), f_V^n\varphi(y), \beta^n\}.
\end{aligned}$$

Thus, $\min\{f_V^n(\varphi(x) \odot_1 \varphi(y)), \alpha^n\} \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y), \beta^n\}$. Similarly,

$$\min\{f_V^n(\varphi(x) \odot_2 \varphi(y)), \alpha^n\} \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y), \beta^n\}$$

and

$$\min\{f_V^n(\varphi(x) \odot_3 \varphi(y)), \alpha^n\} \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y), \beta^n\}.$$

Hence, V is an (α, β)-BVSBS of S_2. □

Theorem 4.8. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings. The homomorphic preimage of (α, β)-BVSBS of S_2 is an (α, β)-BVSBS of S_1.*

Proof: Let $\varphi\colon S_1 \to S_2$ be any homomorphism. Then $\varphi(x \oplus_1 y) = \varphi(x) \odot_1 \varphi(y)$, $\varphi(x \oplus_2 y) = \varphi(x) \odot_2 \varphi(y)$, and $\varphi(x \oplus_3 y) = \varphi(x) \odot_3 \varphi(y)$ for all $x, y \in S_1$. Let $V = \varphi(A)$, where V is any (α, β)-BVSBS of S_2. Let $x, y \in S_1$. Then $\max\{f_A^p(x \oplus_1 y), \alpha^p\} =$

$\max\{f_V^p(\varphi(x \oplus_1 y)), \alpha^p\} = \max\{f_V^p(\varphi(x) \odot_1 \varphi(y)), \alpha^p\} \geq \min\{f_V^p\varphi(x), f_V^p\varphi(y), \beta^p\} = \min\{f_A^p(x), f_A^p(y), \beta^p\}$. Thus, $\max\{f_A^p(x \oplus_1 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$. Similarly,

$$\max\{f_A^p(x \oplus_2 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}$$

and

$$\max\{f_A^p(x \oplus_3 y), \alpha^p\} \geq \min\{f_A^p(x), f_A^p(y), \beta^p\}.$$

Now, $\min\{f_A^n(x \oplus_1 y), \alpha^n\} = \min\{f_V^n(\varphi(x \oplus_1 y)), \alpha^n\} = \min\{f_V^n(\varphi(x) \odot_1 \varphi(y)), \alpha^n\} \leq \max\{f_V^n\varphi(x), f_V^n\varphi(y), \beta^n\} = \max\{f_A^n(x), f_A^n(y), \beta^n\}$. Thus,

$$\min\{f_A^n(x \oplus_1 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}.$$

Similarly,

$$\min\{f_A^n(x \oplus_2 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}$$

and

$$\min\{f_A^n(x \oplus_3 y), \alpha^n\} \leq \max\{f_A^n(x), f_A^n(y), \beta^n\}.$$

Hence, A is an (α, β)-BVSBS of S_1. □

5. (α, β)-Bipolar-Valued Normal Subbisemirings.

In what follows that, let $(\alpha^p, \beta^p) \in [0, 1]$ and $(\alpha^n, \beta^n) \in [-1, 0]$ be such that $0 \leq \alpha^p < \beta^p \leq 1$ and $-1 \leq \beta^n < \alpha^n \leq 0$, both $(\alpha, \beta) \in [0, 1]$ are arbitrary fixed.

Definition 5.1. *An (α, β)-BVSBS A of S is said to be an (α, β)-bipolar-valued normal subbisemiring ((α, β)-BVNSBS) of S if it satisfies the following conditions:*

(1) $f_A^p(x \boxtimes_1 y) = f_A^p(y \boxtimes_1 x)$,
(2) $f_A^p(x \boxtimes_2 y) = f_A^p(y \boxtimes_2 x)$,
(3) $f_A^p(x \boxtimes_3 y) = f_A^p(y \boxtimes_3 x)$,
(4) $f_A^n(x \boxtimes_1 y) = f_A^n(y \boxtimes_1 x)$,
(5) $f_A^n(x \boxtimes_2 y) = f_A^n(y \boxtimes_2 x)$,
(6) $f_A^n(x \boxtimes_3 y) = f_A^n(y \boxtimes_3 x)$, $\forall x, y \in S$.

Theorem 5.1.

(1) The intersection of a family of BVNSBSs of S is a BVNSBS of S.
(2) The intersection of a family of (α, β)-BVNSBSs of S is an (α, β)-BVNSBS of S.

Theorem 5.2.

(1) If $A_1, A_2, \ldots, A_n$ are BVNSBSs of bisemirings $S_1, S_2, \ldots, S_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is a BVNSBS of $S_1 \times S_2 \times \cdots \times S_n$.
(2) If $A_1, A_2, \ldots, A_n$ are (α, β)-BVNSBS of bisemirings $S_1, S_2, \ldots, S_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is an (α, β)-BVNSBS of $S_1 \times S_2 \times \cdots \times S_n$.

Theorem 5.3.

(1) Let A be any BVNSBS of S and V be the strongest bipolar-valued relation on S. Then A is a BVNSBS of S if and only if V is a BVNSBS of $S \times S$.
(2) Let A be any (α, β)-BVNSBS of S and V be the strongest (α, β)-bipolar-valued relation on S. Then A is an (α, β)-BVNSBS of S if and only if V is an (α, β)-BVNSBS of $S \times S$.

Theorem 5.4. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings.*

(1) The homomorphic image of any BVNSBS of S_1 is a BVNSBS of S_2.
(2) The homomorphic image of any (α, β)-BVNSBS of S_1 is an (α, β)-BVNSBS of S_2.

Theorem 5.5. *Let $(S_1, \oplus_1, \oplus_2, \oplus_3)$ and $(S_2, \odot_1, \odot_2, \odot_3)$ be any two bisemirings.*

(1) The homomorphic preimage of any BVNSBS of S_2 *is a BVNSBS of* S_1*.*
(2) The homomorphic preimage of any (α, β)*-BVNSBS of* S_2 *is an* (α, β)*-BVNSBS of* S_1*.*

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