International Journal of Innovative Computing, Information and Control
Volume **19**, Number **3**, June **2023**
ICIC International ©2023 ISSN 1349-4198
pp. 775–787

ON EXTERNAL DIRECT PRODUCTS OF IUP-ALGEBRAS

Chatsuda Chanmanee[1], Rukchart Prasertpong[2], Pongpun Julatha[3]
Nareupanat Lekkoksung[4] and Aiyared Iampan[1,*]

[1]Fuzzy Algebras and Decision-Making Problems Research Unit
Department of Mathematics
School of Science
University of Phayao
19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand
64080930@up.ac.th; *Corresponding author: aiyared.ia@up.ac.th

[2]Division of Mathematics and Statistics
Faculty of Science and Technology
Nakhon Sawan Rajabhat University
398 Moo 9, Tambon Nakhon Sawan Tok, Amphur Mueang, Nakhon Sawan 60000, Thailand
rukchart.p@nsru.ac.th

[3]Department of Mathematics
Faculty of Science and Technology
Pibulsongkram Rajabhat University
156 Moo 5, Tambon Phlai Chumphon, Amphur Mueang, Phitsanulok 65000, Thailand
pongpun.j@psru.ac.th

[4]Division of Mathematics
Faculty of Engineering
Rajamangala University of Technology Isan, Khon Kaen Campus
150 Srichan Road, Amphur Mueang, Khon Kaen 40000, Thailand
nareupanat.le@rmuti.ac.th

Received October 2022; revised January 2023

Abstract. *The concept of the direct product of a finite family of B-algebras is introduced by Lingcong and Endam in 2016. In this paper, we introduce the concept of the direct product of an infinite family of IUP-algebras, we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam, and find the result of the external direct product of special subsets of IUP-algebras. Also, we introduce the concept of the weak direct product of IUP-algebras. Finally, we provide several fundamental theorems of (anti-)IUP-homomorphisms in view of the external direct product IUP-algebras.*
Keywords: IUP-algebra, External direct product, Weak direct product, IUP-homomorphism, Anti-IUP-homomorphism

1. Introduction.

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras and they have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [1, 2]. In 2002, Neggers and Kim [3] constructed a new algebraic structure based on BCI and BCK-algebras and called it a B-algebra. Furthermore, Kim and Kim [4] introduced a new notion, called a BG-algebra which is a generalization of B-algebra. They obtained several isomorphism theorems of BG-algebras and related properties.

DOI: 10.24507/ijicic.19.03.775

Iampan et al. [5] introduced a new algebraic structure, called an independent UP-algebras (in short, IUP-algebras), which is independent of UP-algebras. The concepts of IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals of IUP-algebras play an important role in studying the many logical algebras. In addition, they also discussed the concept of homomorphisms between IUP-algebras and studied the direct and inverse images of four special subsets.

The concept of the direct product [6] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group, and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [7] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras and we obtained some of their properties [8]. In the same year, Endam and Teves [9] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [10] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [11] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [6] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [12] defined the direct product of GK algebra. They derived the finite form of the direct product of GK algebra and function as well. They investigated and applied the concept of the direct product of GK algebra in GK function and GK kernel and obtained interesting results. In 2022, Chanmanee et al. [13] introduced the concept of the direct product of infinite family of B-algebras, and they called the external direct product. Also, they introduced the concept of the weak direct product of B-algebras. Finally, they provided several fundamental theorems of (anti-)B-homomorphisms in view of the external direct product B-algebras. In 2023, Chanmanee et al. [14] introduced the concept of the direct product of infinite family of UP-algebras and proved that it is a DUP-algebras, and they called the external direct product DUP-algebra induced by UP-algebras. They found the result of the external direct product of special subsets of UP-algebras. Also, they introduced the concept of the weak direct product DUP-algebras. Finally, they provided several fundamental theorems of (anti-)UP-homomorphisms in view of the external direct product DUP-algebras. Chanmanee et al. [15] introduced the concept of the direct product of infinite family of UP (BCC)-algebras, they called the external direct product, and found the result of the external direct product of special subsets of UP (BCC)-algebras. Also, they introduced the concept of the weak direct product of UP (BCC)-algebras. Finally, they provided several fundamental theorems of (anti-)UP (BCC)-homomorphisms in view of the external direct product UP (BCC)-algebras. From the reviewed articles, it can be seen that the concept of the direct product is constantly being studied. That inspired us to study it on IUP-algebras parallel to other algebras.

In the next section, we will give a definition of IUP-algebras and introduce the special subsets to be studied and their important properties. In Section 3, our main study results are presented where we introduce the concept of the direct product of infinite family of IUP-algebras, and we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [7]. Moreover, we introduce the concept of the weak direct product of IUP-algebras. Finally, we discuss several (anti-) IUP-homomorphism theorems in view of the external direct product IUP-algebras. In the last section, we summarize the results of the study and will present future research.

2. Preliminaries.

First of all, we start with the definitions and examples of IUP-algebras as well as other relevant definitions for the study in this paper as follows.

Definition 2.1. [5] *An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $*$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 * x = x), \tag{IUP-1}$$

$$(\forall x \in X)(x * x = 0), \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x * y) * (x * z) = y * z). \tag{IUP-3}$$

In an IUP-algebra $X = (X; *, 0)$, the binary relation $\leq$ on X is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	3	2
2	3	5	0	2	1	4
3	2	4	3	0	5	1
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Then $X = (X; *, 0)$ is an IUP-algebra.

In an IUP-algebra $X = (X; *, 0)$, the following assertions are valid (see [5]).

$$(\forall x, y \in X)((x * 0) * (x * y) = y), \tag{1}$$

$$(\forall x \in X)((x * 0) * (x * 0) = 0), \tag{2}$$

$$(\forall x, y \in X)((x * y) * 0 = y * x), \tag{3}$$

$$(\forall x \in X)((x * 0) * 0 = x), \tag{4}$$

$$(\forall x, y \in X)(x * ((x * 0) * y) = y), \tag{5}$$

$$(\forall x, y \in X)(((x * 0) * y) * x = y * 0), \tag{6}$$

$$(\forall x, y, z \in X)(x * y = x * z \Leftrightarrow y = z), \tag{7}$$

$$(\forall x, y \in X)(x * y = 0 \Leftrightarrow x = y), \tag{8}$$

$$(\forall x \in X)(x * 0 = 0 \Leftrightarrow x = 0), \tag{9}$$

$$(\forall x, y, z \in X)(y * x = z * x \Leftrightarrow y = z), \tag{10}$$

$$(\forall x, y \in X)(x * y = y \Rightarrow x = 0), \tag{11}$$

$$(\forall x, y, z \in X)((x * y) * 0 = (z * y) * (z * x)), \tag{12}$$

$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (z * x) * (z * y) = 0), \tag{13}$$

$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (x * z) * (y * z) = 0), \tag{14}$$

the right and the left cancellation laws hold. (15)

Definition 2.2. [5] *A nonempty subset S of an IUP-algebra $X = (X; *, 0)$ is called* (*i*) *an IUP-subalgebra of* X *if it satisfies the following condition:*

$$(\forall x, y \in S)(x * y \in S), \tag{16}$$

(*ii*) *an IUP-filter of* X *if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{17}$$

$$(\forall x, y \in S)(x * y \in S, x \in S \Rightarrow y \in S), \tag{18}$$

(*iii*) *an IUP-ideal of* X *if it satisfies the condition* (17) *and the following condition:*

$$(\forall x, y, z \in S)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{19}$$

(*iv*) *a strong IUP-ideal of* X *if it satisfies the following condition:*

$$(\forall x, y \in S)(y \in S \Rightarrow x * y \in S). \tag{20}$$

From [5], we know that the concept of IUP-filters is a generalization of IUP-ideals and IUP-subalgebras, and IUP-ideals and IUP-subalgebras are a generalization of strong IUP-ideals. In an IUP-algebra X, we have strong IUP-ideals. We get the diagram of the special subsets of IUP-algebras, which is shown in Figure 1.

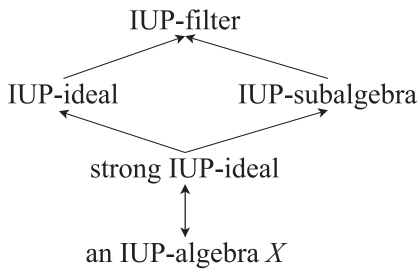

FIGURE 1. Special subsets of IUP-algebras

The concept of IUP-homomorphisms was introduced by Iampan et al. in [5]. Let $X_1 = (X_1; *_1, 0_1)$ and $X_2 = (X_2; *_2, 0_2)$ be IUP-algebras. A map $\psi : X_1 \to X_2$ is called an *IUP-homomorphism* if $\psi(x *_1 y) = \psi(x) *_2 \psi(y)$ for any $x, y \in X_1$. The kernel of ψ denoted by $\ker \psi$ is defined to be the set $\ker \psi = \{x \in X_1 | \psi(x) = 0_2\}$. An IUP-homomorphism ψ is called an IUP-monomorphism, IUP-epimorphism, or IUP-isomorphism if it is one-one, onto, or bijective, respectively. From [5] they prove that kernel of ψ is IUP-subalgebra, IUP-filter, and IUP-ideal.

3. **Main Results.** We have divided this section into 2 subsections: with the first subsection we introduce the concept of the external and weak direct product of IUP-algebras and study their properties related to special subsets of IUP-algebras; in the second subsection, we look at the properties of (anti-)IUP-homomorphisms in terms of the external direct product of IUP-algebras.

3.1. **External and weak direct products.** Lingcong and Endam [7] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated as follows.

Definition 3.1. [7] *Let* $(X_i; *_i)$ *be an algebra for each* $i \in \{1, 2, \ldots, k\}$. *Define the direct product of algebras* $X_1, X_2, \ldots, X_k$ *to be the structure* $\left(\prod_{i=1}^{k} X_i; \otimes\right)$, *where*

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times \cdots \times X_k = \{(x_1, x_2, \ldots, x_k) | x_i \in X_i \ \forall i = 1, 2, \ldots, k\},$$

and whose operation $\otimes$ *is given by*

$$(x_1, x_2, \ldots, x_k) \otimes (y_1, y_2, \ldots, y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, \ldots, x_k *_k y_k)$$

for all $(x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k) \in \prod_{i=1}^{k} X_i$*.*

Now, we extend the concept of the direct product to infinite family of IUP-algebras and provide some of its properties.

Definition 3.2. [15] *Let* X_i *be a nonempty set for each* $i \in I$*. Define the external direct product of sets* X_i *for all* $i \in I$ *to be the set* $\prod_{i \in I} X_i$*, where*

$$\prod_{i \in I} X_i = \left\{ f : I \to \bigcup_{i \in I} X_i \,\middle|\, f(i) \in X_i \ \forall i \in I \right\}.$$

For convenience, we define an element of $\prod_{i \in I} X_i$ *with a function* $(x_i)_{i \in I} : I \to \bigcup_{i \in I} X_i$*, where* $i \mapsto x_i \in X_i$ *for all* $i \in I$*.*

Remark 3.1. [15] *Let* X_i *be a nonempty set and* S_i *a subset of* X_i *for all* $i \in I$*. Then* $\prod_{i \in I} S_i$ *is a nonempty subset of the external direct product* $\prod_{i \in I} X_i$ *if and only if* S_i *is a nonempty subset of* X_i *for all* $i \in I$*.*

Definition 3.3. [15] *Let* $X_i = (X_i; *_i)$ *be an algebra for all* $i \in I$*. Define the binary operation* $\otimes$ *on the external direct product* $\prod_{i \in I} X_i = \left(\prod_{i \in I} X_i; \otimes\right)$ *as follows:*

$$\left(\forall (x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i\right) \left((x_i)_{i \in I} \otimes (y_i)_{i \in I} = (x_i *_i y_i)_{i \in I}\right). \tag{21}$$

Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i \in I} X_i$ as follows:

$$(\forall j \in I) \left(f_{x_i}(j) = \begin{cases} x_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases} \right). \tag{22}$$

Then $f_{x_i} \in \prod_{i \in I} X_i$.

Remark 3.2. *Let* $X_i = (X_i; *_i, 0_i)$ *be an IUP-algebra for all* $i \in I$*. For* $i \in I$*, we have* $f_{0_i} = (0_i)_{i \in I}$*.*

Lemma 3.1. *Let* $X_i = (X_i; *_i, 0_i)$ *be an IUP-algebra for all* $i \in I$*. For* $i \in I$*, let* $x_i, y_i \in X_i$*. Then* $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$*.*

Proof: Now,

$$(\forall j \in I) \left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases} \right).$$

By (IUP-2), we have

$$(\forall j \in I) \left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases} \right).$$

By (22), we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$. $\square$

The following theorem shows that the direct product of IUP-algebras in terms of an infinite family of IUP-algebras is also an IUP-algebra.

Theorem 3.1. *$X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$ is an IUP-algebra, where the binary operation $\otimes$ is defined in Definition 3.3.*

Proof: Assume that $X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$.

(IUP-1) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-1), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus,

$$(0_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(IUP-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-2), we have $x_i *_i x_i = 0_i$ for all $i \in I$. Thus,

$$(x_i)_{i\in I} \otimes (x_i)_{i\in I} = (x_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

(IUP-3) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (IUP-3), we have $(x_i *_i y_i) *_i (x_i *_i z_i) = y_i *_i z_i$ for all $i \in I$. Thus,

$$\begin{aligned}((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I}) &= ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I} \\ &= (y_i *_i z_i)_{i\in I} \\ &= (y_i)_{i\in I} \otimes (z_i)_{i\in I}.\end{aligned}$$

Hence, $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$ is an IUP-algebra.

Conversely, assume that $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$ is an IUP-algebra, where the binary operation $\otimes$ is defined in Definition 3.3. Let $i \in I$.

(IUP-1) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (22). Since $\prod_{i\in I} X_i$ satisfies (IUP-1), we have $(0_i)_{i\in I} \otimes f_{x_i} = f_{x_i}$. By Remark 3.2, Lemma 3.1, and (22), we have $0_i *_i x_i = x_i$.

(IUP-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (22). Since $\prod_{i\in I} X_i$ satisfies (IUP-2), we have $f_{x_i} \otimes f_{x_i} = (0_i)_{i\in I}$. By Lemma 3.1 and (22), we have $x_i *_i x_i = 0_i$.

(IUP-3) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (22). Since $\prod_{i\in I} X_i$ satisfies (IUP-3), we have $(f_{x_i} \otimes f_{y_i}) \otimes (f_{x_i} \otimes f_{z_i}) = f_{y_i} \otimes f_{z_i}$. By Lemma 3.1 and (22), we have $(x_i *_i y_i) *_i (x_i *_i z_i) = y_i *_i z_i$.

Hence, $X_i = (X_i; *_i, 0_i)$ is an IUP-algebra for all $i \in I$. □

We call the IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$ in Theorem 3.1 the external direct product IUP-algebra induced by an IUP-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$. Next, we introduce the concept of the weak direct product of infinite family of IUP-algebras and obtain some of its properties as follows.

Definition 3.4. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Define the weak direct product of an IUP-algebra X_i for all $i \in I$ to be the structure $\prod^{\mathrm{w}}_{i\in I} X_i = \left(\prod^{\mathrm{w}}_{i\in I} X_i; \otimes\right)$, where*

$$\prod^{\mathrm{w}}_{i\in I} X_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, x_i \neq 0_i, \text{ where the number of such } i \text{ is finite} \right\}.$$

Then $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 3.2. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is an IUP-subalgebra of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I | x_i \neq 0_i\}$ and $I_2 = \{i \in I | y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus,

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j y_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (IUP-1), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ y_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus, $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is an IUP-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 3.3. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is an IUP-ideal of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod^{\mathrm{w}}_{i\in I} X_i$ and $(y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I | x_i *_i (y_i *_i z_i) \neq 0_i\}$ and $I_2 = \{i \in I | y_i \neq 0_i\}$ are finite. We shall show that $I_3 \subseteq I_1 \cup I_2$, where $I_3 = \{i \in I | x_i *_i z_i \neq 0_i\}$. Let $j \notin I_1 \cup I_2$. Then $j \notin I_1$ and $j \notin I_2$, so $x_j *_j (y_j *_j z_j) = 0_j$ and $y_j = 0_j$. By (IUP-1), we have $x_j *_j (0_j *_j z_j) = x_j *_j z_j = 0_j$. This implies that $j \notin I_3$, that is, $I_3 \subseteq I_1 \cup I_2$. Since $I_1 \cup I_2$ is finite, we have I_3 is finite. Therefore, $(x_i)_{i\in I} \otimes (z_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is an IUP-ideal of $\prod_{i\in I} X_i$. □

By Theorems 3.2 and 3.3 and Figure 1, we have $\prod^{\mathrm{w}}_{i\in I} X_i$ is an IUP-subalgebra, an IUP-ideal, and an IUP-filter of the external direct product IUP-algebra $\prod_{i\in I} X_i = \big(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\big)$.

Theorem 3.4. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an IUP-subalgebra of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: Assume that S_i is an IUP-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (16), we have $x_i *_i y_i \in S_i$ for all $i \in I$ and so $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an IUP-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an IUP-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (22). By (16) and Lemma 3.1, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (22), we have $x_i *_i y_i \in S_i$. Hence, S_i is an IUP-subalgebra of X_i for all $i \in I$. □

Theorem 3.5. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an IUP-filter of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: Assume that S_i is an IUP-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (18) that $y_i \in S_i$ for all $i \in I$. Thus, $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an IUP-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an IUP-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (22). By Lemma 3.1, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (18), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (22), we have $y_i \in S_i$. Hence, S_i is an IUP-filter of X_i for all $i \in I$. □

Theorem 3.6. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an IUP-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an IUP-ideal of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: Assume that S_i is an IUP-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$, it follows from (19) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus, $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an IUP-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an IUP-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$, and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (22). By Lemma 3.1, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (19) and Lemma 3.1, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (22), we have $x_i *_i z_i \in S_i$. Hence, S_i is an IUP-ideal of X_i for all $i \in I$. □

Theorem 3.7. *Let $X_i = (X_i; *_i, 0_i)$ be an IUP-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong IUP-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong IUP-ideal of the external direct product IUP-algebra $\prod_{i\in I} X_i = \left(\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I}\right)$.*

Proof: Assume that S_i is a strong IUP-ideal of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus, $y_i \in S_i$ for all $i \in I$, it follows from (20) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus, $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong IUP-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong IUP-ideal of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (22). By (20) and Lemma 3.1, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (22), we have $x_i *_i y_i \in S_i$. Hence, S_i is a strong IUP-ideal of X_i for all $i \in I$. □

3.2. **(Anti-)IUP-homomorphisms.** In this subsection, we provide some properties of (anti-)IUP-homomorphisms in view of the external direct product of IUP-algebras.

Definition 3.5. [13] *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by*

$$\left(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i\right) (\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}). \tag{23}$$

Theorem 3.8. [13] *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 3.5,

(ii) ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,

(iii) ψ_i *is bijective for all* $i \in I$ *if and only if* ψ *is bijective.*

At this point, we provide the essential properties of IUP-homomorphisms in view of the external direct product of IUP-algebras.

Theorem 3.9. *Let* $X_i = (X_i; *_i, 0_i)$ *and* $S_i = (S_i; \circ_i, 1_i)$ *be IUP-algebras and* $\psi_i : X_i \to S_i$ *be a function for all* $i \in I$*. Then*

(i) ψ_i *is an IUP-homomorphism for all* $i \in I$ *if and only if* ψ *is an IUP-homomorphism which is defined in Definition 3.5,*

(ii) ψ_i *is an IUP-monomorphism for all* $i \in I$ *if and only if* ψ *is an IUP-monomorphism,*

(iii) ψ_i *is an IUP-epimorphism for all* $i \in I$ *if and only if* ψ *is an IUP-epimorphism,*

(iv) ψ_i *is an IUP-isomorphism for all* $i \in I$ *if and only if* ψ *is an IUP-isomorphism,*

(v) $\ker \psi = \prod_{i\in I} \ker \psi_i$ *and* $\psi\left(\prod_{i\in I} X_i\right) = \prod_{i\in I} \psi_i(X_i)$.

Proof: *(i)* Assume that ψ_i is an IUP-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I} \\
&= (\psi_i(x_i *_i x'_i))_{i\in I} \\
&= (\psi_i(x_i) *_i \psi_i(x'_i))_{i\in I} \\
&= (\psi_i(x_i))_{i\in I} \otimes (\psi_i(x'_i))_{i\in I} \\
&= \psi(x_i)_{i\in I} \otimes \psi(x'_i)_{i\in I}.
\end{aligned}$$

Hence, ψ is an IUP-homomorphism.

Conversely, assume that ψ is an IUP-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (22). Since ψ is an IUP-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{x_i}) \otimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{24}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \otimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{25}$$

By (24) and (25), we have $\psi_i(x_i *_i y_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is an IUP-homomorphism for all $i \in I$.

(ii) It is straightforward from *(i)* and Theorem 3.8 *(i)*.

(iii) It is straightforward from *(i)* and Theorem 3.8 *(ii)*.

(iv) It is straightforward from *(i)* and Theorem 3.8 *(iii)*.

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I \\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I \\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}(y_i)_{i\in I} \in \psi\left(\prod_{i\in I} X_i\right) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I} \\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I} \\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I \\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).\end{aligned}$$

Hence, $\psi\left(\prod_{i\in I} X_i\right) = \prod_{i\in I} \psi_i(X_i)$. □

Finally, we provide the essential properties of anti-IUP-homomorphisms in view of the external direct product of IUP-algebras.

Theorem 3.10. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be IUP-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) ψ_i is an anti-IUP-homomorphism for all $i \in I$ if and only if ψ is an anti-IUP-homomorphism which is defined in Definition 3.5,

(ii) ψ_i is an anti-IUP-monomorphism for all $i \in I$ if and only if ψ is an anti-IUP-monomorphism,

(iii) ψ_i is an anti-IUP-epimorphism for all $i \in I$ if and only if ψ is an anti-IUP-epimorphism,

(iv) ψ_i is an anti-IUP-isomorphism for all $i \in I$ if and only if ψ is an anti-IUP-isomorphism.

Proof: (i) Assume that ψ_i is an anti-IUP-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}$, $(x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I} \\ &= (\psi_i(x_i *_i x'_i))_{i\in I} \\ &= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I} \\ &= (\psi_i(x'_i))_{i\in I} \otimes (\psi_i(x_i))_{i\in I} \\ &= \psi(x'_i)_{i\in I} \otimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-IUP-homomorphism.

Conversely, assume that ψ is an anti-IUP-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (22). Since ψ is an anti-IUP-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{y_i}) \otimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right),$$

we have

$$(\forall j \in I) \left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases} \right). \tag{26}$$

Since

$$(\forall j \in I) \left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right)$$

and

$$(\forall j \in I) \left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right),$$

we have

$$(\forall j \in I) \left((\psi(f_{y_i}) \otimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases} \right). \tag{27}$$

By (26) and (27), we have $\psi_i(x_i *_i y_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an anti-IUP-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 3.8 (i).

(iii) It is straightforward from (i) and Theorem 3.8 (ii).

(iv) It is straightforward from (i) and Theorem 3.8 (iii). □

4. **Conclusions and Future Work.** In this paper, we have introduced the concept of the direct product of infinite family of IUP-algebras, and we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [7]. We proved that the external direct product of IUP-algebras is also an IUP-algebra. Also, we have introduced the concept of the weak direct product of IUP-algebras. We proved that the weak direct product of IUP-algebras is an IUP-subalgebra, IUP-ideal, and IUP-filter, and the external direct product of IUP-subalgebras (resp., IUP-filters, IUP-ideals, strong IUP-ideals) is also an IUP-subalgebra (resp., IUP-filter, IUP-ideal, strong IUP-ideal) of the external direct product IUP-algebras. Finally, we have provided several fundamental theorems of (anti-)IUP-homomorphisms in view of the external direct product IUP-algebras.

Based on the concept of the external direct product of IUP-algebras in this article, we can apply it to the study of the external direct product in other algebraic systems. A novel idea for the study of the internal direct product of IUP-algebras will be created in the near future based on the results of the external direct product of IUP-algebras in this work. The essential IUP-ideals and essential IUP-filters presented by Gaketem et al. [16, 17] will also be included in our expanded analysis of the external direct product of IUP-algebras.

Acknowledgment. This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF66-UoE017).

REFERENCES

[1] Y. Imai and K. Iséki, On axiom systems of propositional calculi, XIV, *Proc. of Japan Acad.*, vol.42, no.1, pp.19-22, 1966.
[2] K. Iséki, An algebra related with a propositional calculus, *Proc. of Japan Acad.*, vol.42, no.1, pp.26-29, 1966.
[3] J. Neggers and H. S. Kim, On B-algebras, *Mat. Vesnik*, vol.54, pp.21-29, 2002.
[4] C. B. Kim and H. S. Kim, On BG-algebras, *Demonstr. Math.*, vol.61, no.3, pp.497-505, 2008.

[5] A. Iampan, P. Khamrot, P. Julatha and D. A. Romano, Independent UP-algebras, *J. Math. Comput. Sci. (JMCS)*, vol.27, no.1, pp.65-76, 2022.
[6] A. Setiani, S. Gemawati and L. Deswita, Direct product of BP-algebra, *Int. J. Math. Trends Technol.*, vol.66, no.2, pp.63-66, 2020.
[7] J. A. V. Lingcong and J. C. Endam, Direct product of B-algebras, *Int. J. Algebra*, vol.10, no.1, pp.33-40, 2016.
[8] J. A. V. Lingcong and J. C. Endam, Mappings of the direct product of B-algebras, *Int. J. Algebra*, vol.10, no.3, pp.133-140, 2016.
[9] J. C. Endam and R. C. Teves, Direct product of BF-algebras, *Int. J. Algebra*, vol.10, no.3, pp.125-132, 2016.
[10] G. A. Abebe, *On the Theory of BRK-Algebras*, Ph.D. Thesis, Department of Mathematics, Addis Ababa University, 2018.
[11] S. Widianto, S. Gemawati and Kartini, Direct product in BG-algebras, *Int. J. Algebra*, vol.13, no.5, pp.239-247, 2019.
[12] J. Kavitha and R. Gowri, Direct product of GK algebra, *Indian J. Technol.*, vol.14, no.35, pp.2802-2805, 2021.
[13] C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha and A. Iampan, Direct product of infinite family of B-algebras, *Eur. J. Pure Appl. Math.*, vol.15, no.3, pp.999-1014, 2022.
[14] C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha and A. Iampan, External direct products on dual UP (BCC)-algebras, *J. Math. Comput. Sci. (JMCS)*, vol.29, no.2, pp.175-191, 2023.
[15] C. Chanmanee, R. Chinram, N. Phrommarat, P. Julatha, N. Lekkoksung and A. Iampan, External and weak direct products of UP (BCC)-algebras, *J. Math. Comput. Sci. (JMCS)*, vol.29, no.1, pp.90-105, 2023.
[16] T. Gaketem and A. Iampan, Essential UP-ideals and t-essential fuzzy UP-ideals of UP-algebras, *ICIC Express Letters*, vol.15, no.12, pp.1283-1289, 2021.
[17] T. Gaketem, P. Khamrot and A. Iampan, Essential UP-filters and t-essential fuzzy UP-filters of UP-algebras, *ICIC Express Letters*, vol.16, no.10, pp.1057-1062, 2022.

Author Biography

Chatsuda Chanmanee is an M.Sc. student at Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand, under the thesis advisor of Associate Professor Dr. Aiyared Iampan. She received her B.Sc. and B.Ed. degrees in Mathematics from Kamphaeng Phet Rajabhat University, Kamphaeng Phet, Thailand. Her areas of interest include fuzzy algebraic structures and logical algebras.

Rukchart Prasertpong is a faculty member of the Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Thailand. His areas of interest include algebraic theory of semigroups and Γ-semigroups, fuzzy algebraic structures, and rough algebraic structures.

Pongpun Julatha is a faculty member of the Faculty of Science and Technology, Pibulsongkram Rajabhat University, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Thailand. His areas of interest include algebraic theory of semigroups, ternary semigroups, and Γ-semigroups and fuzzy algebraic structures.

Nareupanat Lekkoksung obtained his B.S., M.S., and Ph.D. from Khon Kaen University, under the supervision of Assistant Professor Prakit Jampachon. He is currently Assistant Professor of Rajamangala University of Technology Isan, Khon Kaen Campus, since 2020. His research interests are hypersubstitution theory and various algebraic structures theories such as ordered semigroups and ordered semihypergroups.

Aiyared Iampan is an Associate Professor at Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Phitsanulok, Thailand, under the thesis advisor of Professor Dr. Manoj Siripitukdet. His areas of interest include algebraic theory of semigroups, ternary semigroups, and Γ-semigroups, lattices and ordered algebraic structures, fuzzy algebraic structures, and logical algebras. He was the founder of the Group for Young Algebraists in University of Phayao in 2012 and one of the co-founders of the Fuzzy Algebras and Decision-Making Problems Research Unit in University of Phayao in 2021.