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INTUITIONISTIC HESITANT FUZZY SUBALGEBRAS AND IDEALS OF HILBERT ALGEBRAS

Aiyared Iampan[1,*], Ramasamy Subasini[2]
Ponnusamy Maragatha Meenakshi[3] and Neelamegarajan Rajesh[4]

[1]Fuzzy Algebras and Decision-Making Problems Research Unit
Department of Mathematics
School of Science
University of Phayao
19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand
*Corresponding author: aiyared.ia@up.ac.th

[2]Department of Mathematics
Pollachi Institute of Engineering and Technology
Tamilnadu, Pollachi 642205, India
subasinimaths@gmail.com

[3]Department of Mathematics
Thanthai Periyar Government Arts and Science College (affiliated to Bharathidasan University)
Tamilnadu, Tiruchirappalli 620023, India
maragathameenakship@gmail.com

[4]Department of Mathematics
Rajah Serfoji Government College (affiliated to Bharathidasan University)
Tamilnadu, Thanjavur 613005, India
nrajesh_topology@yahoo.co.in

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Abstract. *The concepts of intuitionistic hesitant fuzzy subalgebras and ideals of Hilbert algebras are introduced, some of their features are defined, and their extensions are established using the theory of hesitant fuzzy sets (HFSs) as a foundation. We also discuss the link between these intuitionistic hesitant fuzzy sets (IHFSs) and their complement. It is also explored how intuitionistic hesitant fuzzy subalgebras (ideals) relate to their* π*-level subsets. Hilbert algebras are also investigated in terms of the homomorphic pre-images of intuitionistic hesitant fuzzy subalgebras (ideals) and other related properties.*
Keywords: Hilbert algebra, Intuitionistic hesitant fuzzy set, Intuitionistic hesitant fuzzy subalgebra, Intuitionistic hesitant fuzzy ideal, π-level subset, Homomorphic pre-image

1. **Introduction.** The concept of fuzzy sets was proposed by Zadeh [1]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets, one of which is the intuitionistic fuzzy set defined by Atanassov [2]. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [3, 4, 5, 6]. In 2009-2010, Torra and Narukawa [7, 8] introduced the notion of hesitant fuzzy sets, that is a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion fuzzy sets. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of hesitant fuzzy sets

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by Torra and Narukawa [7, 8], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application to many logical algebras. For example, in 2012, Zhu et al. [9] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun et al. [10] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [11] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2015, Jun and Song [12] introduced the notions of hesitant fuzzy prefilters (resp., filters) and positive implicative hesitant fuzzy prefilters (resp., filters) of EQ-algebras. In 2016, Jun and Ahn [13] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Iampan [14] introduced a new algebraic structure, called a UP-algebra, Somjanta et al. [15] introduced the notion of fuzzy sets in UP-algebras, and Mosrijai et al. [16] introduced the notion of hesitant fuzzy sets on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [17] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [18] from algebraic point of view. Diego [18] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [19, 20] and Jun [21] and some of their filters forming deductive systems were recognized. Dudek [22, 23, 24] considered the fuzzification of subalgebras, ideals and deductive systems in Hilbert algebras. The concepts of subalgebras and ideals in Hilbert algebras and semigroups are studied in many dimensions; for example, Iampan et al. [25, 26] introduced and studied anti-hesitant fuzzy subalgebras and ideals, and interval-valued intuitionistic fuzzy subalgebras and ideals of Hilbert algebras in 2022. This year they also introduced and studied interval-valued neutrosophic subalgebras and ideals of Hilbert algebras [27, 28]. Moreover, they also studied fuzzy subalgebras and ideals with thresholds of Hilbert algebras [29]. Phummee et al. [30] introduced and studied the concept of sup-hesitant fuzzy interior ideals of semigroups. In 2023, Iampan et al. [31, 32] introduced and studied anti-fuzzy subalgebras and ideals, and intuitionistic fuzzy subalgebras and ideals of Hilbert algebras. This year they also introduced and studied intuitionistic $\mathcal{N}$-fuzzy subalgebras and ideals of Hilbert algebras [33]. Based on the concept of subalgebras and ideals, we are interested in studying them from the point of view of intuitionistic hesitant fuzzy sets.

This study builds on the theory of HFSs to introduce the notions of intuitionistic hesitant fuzzy subalgebras and ideals of Hilbert algebras, describe some of their properties, and prove some of their extensions. The relationship between these IHFSs and their complement is also discussed. Additionally, the relationship between intuitionistic hesitant fuzzy subalgebras (ideals) and their π-level subsets is examined. The homomorphic preimages of hesitant intuitionistic fuzzy subalgebras (ideals) and other associated features are also examined in relation to Hilbert algebras.

2. **Preliminaries.** The concept of Hilbert algebras as it was initially introduced by Diego [18] in 1966 will be reviewed initially.

Definition 2.1. [18] *A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [22], the following conclusion was established.

Lemma 2.1. *Let* $X = (X, \cdot, 1)$ *be a Hilbert algebra. Then*
(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. [34] *A nonempty subset* D *of a Hilbert algebra* $X = (X, \cdot, 1)$ *is called a subalgebra of* X *if* $x \cdot y \in D$ *for all* $x, y \in D$.

Definition 2.3. [35] *A nonempty subset* D *of a Hilbert algebra* $X = (X, \cdot, 1)$ *is called an ideal of* X *if the following conditions hold:*
(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4. [7] *A hesitant fuzzy set (HFS) on a reference set* X *is defined in term of a function* h *that when applied to* X *return a subset of* $[0, 1]$, *that is,* $h : X \to \mathcal{P}([0, 1])$.

Definition 2.5. [36] *An intuitionistic hesitant fuzzy set (IHFS) on a reference set* X *is defined in the form* $\mathcal{H} = (h, k)$, *where* h *and* k *are functions that when applied to* X *return a subset of* $[0, 1]$, *that is,* $h, k : X \to \mathcal{P}([0, 1])$.

Definition 2.6. [7] *The complement of an HFS* h *in a reference set* X *is the HFS* $\overline{h}$ *defined by* $\overline{h}(x) = [0, 1] - h(x)$ *for all* $x \in X$.

Definition 2.7. [7] *The complement of an IHFS* $\mathcal{H} = (h, k)$ *on a reference set* X *is the IHFS* $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ *defined by* $\overline{h}(x) = [0, 1] - h(x)$ *and* $\overline{k}(x) = [0, 1] - k(x)$ *for all* $x \in X$.

Definition 2.8. [37] *An HFS* h *on a Hilbert algebra* $X = (X, \cdot, 1)$ *is said to be*
(1) a hesitant fuzzy subalgebra of X *if it satisfies the following property:*

$$(\forall x, y \in X)\left(\ h(x \cdot y) \supseteq h(x) \cap h(y) \ \right), \tag{1}$$

(2) a hesitant fuzzy ideal of X *if the following conditions hold:*

$$(\forall x \in X)\left(\ h(1) \supseteq h(x) \ \right), \tag{2}$$

$$(\forall x, y \in X)\left(\ h(x \cdot y) \supseteq h(y) \ \right), \tag{3}$$

$$(\forall x, y_1, y_2 \in X)\left(\ h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2) \ \right). \tag{4}$$

3. **Intuitionistic Hesitant Fuzzy Subalgebras and Ideals.** In this section, we introduce the concepts of intuitionistic hesitant fuzzy subalgebras and ideals of Hilbert algebras and provide some interesting properties.

Definition 3.1. *An IHFS* $\mathcal{H} = (h, k)$ *on a Hilbert algebra* $X = (X, \cdot, 1)$ *is called an intuitionistic hesitant fuzzy subalgebra of* X *if it satisfies the following property:*

$$(\forall x, y \in X)\left(\begin{array}{c} h(x \cdot y) \supseteq h(x) \cap h(y) \\ k(x \cdot y) \subseteq k(x) \cup k(y) \end{array} \right). \tag{5}$$

Example 3.1. *Let $X = \{1, a, b, c, d\}$ with the following Cayley table:*

$\cdot$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Then X is a Hilbert algebra. We define an IHFS $\mathcal{H} = (h, k)$ on X as follows:

$$h(1) = \{0.5, 0.2\}, \quad h(a) = h(b) = h(c) = h(d) = \{0.2\},$$
$$k(1) = \{0.2\}, \quad k(a) = k(b) = k(c) = k(d) = \{0.2, 0.5\}.$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X.

Proposition 3.1. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then the following property holds:*

$$(\forall x \in X)\left(\begin{array}{c} h(1) \supseteq h(x) \\ k(1) \subseteq k(x) \end{array}\right). \tag{6}$$

Proof: For any $x \in X$, we have

$$h(1) = h(x \cdot x) \supseteq h(x) \cap h(x) = h(x),$$
$$k(1) = k(x \cdot x) \subseteq k(x) \cup k(x) = k(x). \qquad \square$$

Definition 3.2. *The characteristic intuitionistic hesitant fuzzy set (CIHFS) of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A})$, where*

$$h_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases} \text{ and } k_{\chi_A}(x) = \begin{cases} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise.} \end{cases}$$

Lemma 3.1. *The constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset B of X if and only if $h_{\chi_B}(1) \supseteq h_{\chi_B}(x)$ and $k_{\chi_B}(1) \subseteq k_{\chi_B}(x)$ for all $x \in X$.*

Proof: If $1 \in B$, then $h_{\chi_B}(1) = [0, 1]$. Thus, $h_{\chi_B}(1) = [0, 1] \supseteq h_{\chi_B}(x)$ for all $x \in X$. Also, $k_{\chi_B}(1) = \emptyset$. Then $k_{\chi_B}(1) = \emptyset \subseteq k_{\chi_B}(x)$ for all $x \in X$.

Conversely, assume that $h_{\chi_B}(1) \supseteq h_{\chi_B}(x)$ and $k_{\chi_B}(1) \subseteq k_{\chi_B}(x)$ for all $x \in X$. Since B is a nonempty subset of X, we have $a \in B$ for some $a \in X$. Then $h_{\chi_B}(1) \supseteq h_{\chi_B}(a) = [0, 1]$, so $h_{\chi_B}(1) = [0, 1]$. Hence, $1 \in B$. $\square$

Theorem 3.1. *A nonempty subset S of a Hilbert algebra $X = (X, \cdot, 1)$ is a subalgebra of X if and only if the CIHFS χ_S is an intuitionistic hesitant fuzzy subalgebra of X.*

Proof: Assume that S is a subalgebra of X. Let $x, y \in X$.

Case 1: If $x, y \in S$, then $h_{\chi_S}(x) = [0, 1]$ and $h_{\chi_S}(y) = [0, 1]$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0, 1]$. Since S is a subalgebra of X, $x \cdot y \in S$ and so $h_{\chi_S}(x \cdot y) = [0, 1]$. Then $h_{\chi_S}(x \cdot y) = [0, 1] \supseteq [0, 1] = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = \emptyset$. Thus, $k_{\chi_S}(x) \cup k_{\chi_S}(y) = \emptyset$. Since S is a subalgebra of X, $x \cdot y \in S$ and so $k_{\chi_S}(x \cdot y) = \emptyset$. Then $k_{\chi_S}(x \cdot y) = \emptyset \subseteq \emptyset = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 2: If $x \in S$ and $y \notin S$, then $h_{\chi_S}(x) = [0, 1]$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = [0, 1]$. Thus, $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0, 1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0, 1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 3: If $x \notin S$ and $y \in S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = [0, 1]$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Then $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0, 1]$ and $k_{\chi_S}(y) = \emptyset$. Thus, $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0, 1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0, 1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Case 4: If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus, $h_{\chi_S}(x) \cap h_{\chi_S}(y) = \emptyset$. Hence, $h_{\chi_S}(x \cdot y) \supseteq \emptyset = h_{\chi_S}(x) \cap h_{\chi_S}(y)$. Also, $k_{\chi_S}(x) = [0,1]$ and $k_{\chi_S}(y) = [0,1]$. Thus, $k_{\chi_S}(x) \cup k_{\chi_S}(y) = [0,1]$. Then $k_{\chi_S}(x \cdot y) \subseteq [0,1] = k_{\chi_S}(x) \cup k_{\chi_S}(y)$.

Hence, χ_S is an intuitionistic hesitant fuzzy subalgebra of X.

Conversely, assume that χ_S is an intuitionistic hesitant fuzzy subalgebra of X. Let $x, y \in S$. Then $h_{\chi_S}(x) = [0,1]$ and $h_{\chi_S}(y) = [0,1]$. Thus, $h_{\chi_S}(x \cdot y) \supseteq h_{\chi_S}(x) \cap h_{\chi_S}(y) = [0,1]$, so $h_{\chi_S}(x \cdot y) = [0,1]$. Hence, $x \cdot y \in S$, that is, S is a subalgebra of X. □

Definition 3.3. *An IHFS $\mathcal{H} = (h,k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is called an intuitionistic hesitant fuzzy ideal of X if (6) and the following conditions hold:*

$$(\forall x, y \in X) \left(\begin{array}{c} h(x \cdot y) \supseteq h(y) \\ k(x \cdot y) \subseteq k(y) \end{array} \right), \tag{7}$$

$$(\forall x, y_1, y_2 \in X) \left(\begin{array}{c} h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2) \\ k((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq k(y_1) \cup k(y_2) \end{array} \right). \tag{8}$$

Example 3.2. *Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an IHFS $\mathcal{H} = (h,k)$ on X as follows:

$$h(1) = \{0.4, 0.5, 0.7\}, \quad h(x) = \{0.4, 0.5\}, \quad h(y) = \{0.5\}, \quad h(z) = h(0) = \emptyset,$$
$$k(1) = \emptyset, \quad k(x) = k(y) = k(z) = k(0) = \{0.4, 0.5, 0.7\}.$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy ideal of X.

Proposition 3.2. *If $\mathcal{H} = (h,k)$ is intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then the following property holds:*

$$(\forall x, y \in X) \left(\begin{array}{c} h((y \cdot x) \cdot x) \supseteq h(y) \\ k((y \cdot x) \cdot x) \subseteq k(y) \end{array} \right). \tag{9}$$

Proof: Putting $y_1 = y$ and $y_2 = 1$ in (8), we have

$$h((y \cdot x) \cdot x) \supseteq h(y) \cap h(1) = h(y),$$
$$k((y \cdot x) \cdot x) \subseteq k(y) \cup k(1) = k(y).$$

□

Lemma 3.2. *If $\mathcal{H} = (h,k)$ is an intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then the following property holds:*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \left\{ \begin{array}{c} h(x) \subseteq h(y) \\ k(x) \supseteq k(y) \end{array} \right. \right). \tag{10}$$

Proof: Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned} h(y) &= h(1 \cdot y) \\ &= h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\supseteq h(x \cdot y) \cap h(x) \\ &= h(1) \cap h(x) \\ &= h(x), \end{aligned}$$

$$\begin{aligned} k(y) &= k(1 \cdot y) \\ &= k(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\subseteq k(x \cdot y) \cup k(x) \\ &= k(1) \cup k(x) \\ &= k(x). \end{aligned}$$ □

Theorem 3.2. *Every intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an intuitionistic hesitant fuzzy subalgebra.*

Proof: Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy ideal of X. Let $x, y \in X$. It follows from (7) that

$$h(x \cdot y) \supseteq h(y) \supseteq h(x) \cap h(y),$$
$$k(x \cdot y) \subseteq k(y) \subseteq k(x) \cup k(y).$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.2 is not true in general.

Example 3.3. *Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then X is a Hilbert algebra. We define an IHFS $\mathcal{H}$ on X as follows:

$$h(0) = [0, 1], \quad h(1) = \{0.7\}, \quad h(2) = \emptyset, \quad h(3) = \{0.2, 0.5\},$$
$$k(0) = \emptyset, \quad k(1) = \{0.1, 0.2\}, \quad k(2) = \{0.1, 0.2, 0.3\}, \quad k(3) = [0, 1].$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X but not intuitionistic hesitant fuzzy ideal of X.

The following theorem can be proved similarly to Theorem 3.1.

Theorem 3.3. *A nonempty subset S of a Hilbert algebra $X = (X, \cdot, 1)$ is an ideal of X if and only if the CIHFS χ_S is an intuitionistic hesitant fuzzy ideal of X.*

Definition 3.4. *Let $\mathcal{H} = (h, k)$ be an IHFS on a set X. The IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are defined as $\oplus\mathcal{H} = \left(h, \overline{h}\right)$ and $\otimes\mathcal{H} = \left(\overline{k}, k\right)$.*

Theorem 3.4. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy subalgebras of X.*

Proof: Let $x, y \in X$. Then

$$\begin{aligned} \overline{h}(x \cdot y) &= [0, 1] - h(x \cdot y) \\ &\subseteq [0, 1] - (h(x) \cap h(y)) \\ &= ([0, 1] - h(x)) \cup ([0, 1] - h(y)) \\ &= \overline{h}(x) \cup \overline{h}(y). \end{aligned}$$

Hence, $\oplus\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. Let $x, y \in X$. Then

$$\begin{aligned} \overline{k}(x \cdot y) &= [0, 1] - k(x \cdot y) \\ &\supseteq [0, 1] - (k(x) \cup k(y)) \\ &= ([0, 1] - k(x)) \cap ([0, 1] - k(y)) \\ &= \overline{k}(x) \cap \overline{k}(y). \end{aligned}$$

Hence, $\otimes\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X.

Conversely, assume that $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy subalgebras of X. Then for any $x, y \in X$, we have $h(x \cdot y) \supseteq h(x) \cap h(y)$ and $k(x \cdot y) \subseteq k(x) \cup k(y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. □

Theorem 3.5. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then the sets $X_h = \{x \in X \mid h(x) = h(1)\}$ and $X_k = \{x \in X \mid k(x) = k(1)\}$ are subalgebras of X.*

Proof: Let $x, y \in X_h$. Then $h(x) = h(1) = h(y)$ and so $h(x \cdot y) \supseteq h(x) \cap h(y) = h(1)$. By using Proposition 3.1, we have $h(x \cdot y) = h(1)$; hence $x \cdot y \in X_h$. Again, let $x, y \in X_k$. Then $k(x) = k(1) = k(y)$ and so $k(x \cdot y) \subseteq k(x) \cup k(y) = k(1)$. Again, by Proposition 3.1, we have $k(x \cdot y) = k(1)$; hence $x \cdot y \in X_k$. Hence, X_h and X_k are subalgebras of X. □

The following theorem can be proved similarly to Theorem 3.4.

Theorem 3.6. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy ideals of X.*

The following theorem can be proved similarly to Theorem 3.5.

Theorem 3.7. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then the sets $X_h = \{x \in X \mid h(x) = h(1)\}$ and $X_k = \{x \in X \mid k(x) = k(1)\}$ are ideals of X.*

Theorem 3.8. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy subalgebras of X.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. Then for any $x, y \in X, h(x \cdot y) \supseteq h(x) \cap h(y)$. Hence, h is a hesitant fuzzy subalgebra of X. Now for any $x, y \in X$, we have

$$\begin{aligned}\overline{k}(x \cdot y) &= [0,1] - k(x \cdot y) \\ &\supseteq [0,1] - (k(x) \cup k(y)) \\ &= [0,1] - k(x) \cap [0,1] - k(y) \\ &= \overline{k}(x) \cap \overline{k}(y).\end{aligned}$$

Hence, $\overline{k}$ is a hesitant fuzzy subalgebra of X.

Conversely, assume that the HFSs h and $\overline{k}$ are hesitant fuzzy subalgebras of X. Then for any $x, y \in X, h(x \cdot y) \supseteq h(x) \cap h(y)$. Now for any $x, y \in X$, we have $\overline{k}(x \cdot y) \supseteq \overline{k}(x) \cap \overline{k}(y)$. Then

$$\begin{aligned}[0,1] - k(x \cdot y) &\supseteq [0,1] - k(x) \cap [0,1] - k(y) \\ &= [0,1] - (k(x) \cup k(y)), \\ k(x \cdot y) &\subseteq k(x) \cup k(y).\end{aligned}$$

Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. □

The following theorem can be proved similarly to Theorem 3.8.

Theorem 3.9. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy ideals of X.*

Theorem 3.10. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFS $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ is an intuitionistic hesitant fuzzy ideal of X.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic fuzzy ideal of X. Then for any $x, y, y_1, y_2 \in X$, we have $h(1) \supseteq h(x)$, $h(x \cdot y) \supseteq h(y)$ and $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)$. Hence, for any $x, y, y_1, y_2 \in X$, we have $\overline{h}(1) = [0,1] - h(1) \subseteq [0,1] - h(x) = \overline{h}(x)$, $\overline{h}(x \cdot y) = [0,1] - h(x \cdot y) \subseteq [0,1] - h(y) = \overline{h}(y)$ and

$$\begin{aligned}\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= [0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\subseteq [0,1] - (h(y_1) \cap h(y_2)) \\ &= [0,1] - h(y_1) \cup [0,1] - h(y_2) \\ &= \overline{h}(y_1) \cup \overline{h}(y_2).\end{aligned}$$

Now, for any $x, y, y_1, y_2 \in X$, we have $k(1) \subseteq k(x)$, $k(x \cdot y) \subseteq k(y)$ and $k((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq k(y_1) \cup k(y_2)$. Hence, for any $x, y, y_1, y_2 \in X$, we have $\overline{k}(1) = [0,1] - k(1) \supseteq [0,1] - k(x) = \overline{k}(x)$, $\overline{k}(x \cdot y) = [0,1] - k(x \cdot y) \supseteq [0,1] - k(y) = \overline{k}(y)$ and

$$\begin{aligned}\overline{k}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= [0,1] - k((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\supseteq [0,1] - (k(y_1) \cup k(y_2)) \\ &= [0,1] - k(y_1) \cap [0,1] - k(y_2) \\ &= \overline{k}(y_1) \cap \overline{k}(y_2).\end{aligned}$$

Hence, $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy ideal of X.

Conversely, assume that the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy ideal of X. Then for any $x, y, y_1, y_2 \in X$, we have $\overline{k}(1) \supseteq \overline{k}(x)$, $\overline{k}(x \cdot y) \supseteq \overline{k}(y)$ and $\overline{k}((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq \overline{k}(y_1) \cap \overline{k}(y_2)$. Then $[0,1] - k(1) \supseteq [0,1] - k(x)$, $[0,1] - k(x \cdot y) \supseteq [0,1] - k(y)$ and $[0,1] - k((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq [0,1] - (k(y_1) \cup k(y_2))$, so $k(1) \subseteq k(x)$, $k(x \cdot y) \subseteq k(y)$ and $k((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq k(y_1) \cup k(y_2)$. Now, for any $x, y, y_1, y_2 \in X$, we have $\overline{h}(1) \subseteq \overline{h}(x)$, $\overline{h}(x \cdot y) \subseteq \overline{h}(y)$ and $\overline{h}((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq \overline{h}(y_1) \cup \overline{h}(y_2)$. Then $[0,1] - h(1) \subseteq [0,1] - h(x)$, $[0,1] - h(x \cdot y) \subseteq [0,1] - h(y)$ and $[0,1] - h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq [0,1] - (h(y_1) \cup h(y_2))$, so $h(1) \supseteq h(x)$, $h(x \cdot y) \supseteq h(y)$ and $h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy ideal of X. □

The following theorem can be proved similarly to Theorem 3.10.

Theorem 3.11. *An IHFS $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFS $\overline{\mathcal{H}} = (\overline{k}, \overline{h})$ is an intuitionistic hesitant fuzzy subalgebra of X.*

Definition 3.5. *Let $h : X \to \mathcal{P}([0,1])$. For any $\pi \in \mathcal{P}([0,1])$, the sets $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h, \pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h, \pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h, \pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then $U(h, \pi) = U^+(h, \pi) \cup E(h, \pi)$ and $L(h, \pi) = L^-(h, \pi) \cup E(h, \pi)$.*

Theorem 3.12. *An IHFS $\mathcal{H} = (h, k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is an intuitionistic hesitant fuzzy subalgebra of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are subalgebras.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h, \pi) \neq \emptyset$ and let $x, y \in U(h, \pi)$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X, we have $h(x \cdot y) \supseteq h(x) \cap h(y) \supseteq \pi$ and thus $x \cdot y \in U(h, \pi)$. So, $U(h, \pi)$ is a subalgebra of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(k, \pi) \neq \emptyset$ and let $x, y \in L(k, \pi)$. Then $k(x) \subseteq \pi$ and $k(y) \subseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X, we have $k(x \cdot y) \subseteq k(x) \cup k(y) \subseteq \pi$ and thus $x \cdot y \in L(k, \pi)$. So, $L(k, \pi)$ is a subalgebra of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are subalgebras of X. Let $x, y \in X$. Choose $\pi = h(x) \cap h(y) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$ and $h(y) \supseteq \pi$. Thus, $x, y \in U(h, \pi) \neq \emptyset$. By assumption, $U(h, \pi)$ is a subalgebra of X and thus $x \cdot y \in U(h, \pi)$. So, $h(x \cdot y) \supseteq \pi = h(x) \cap h(y)$. Let $x, y \in X$. Choose $\pi_1 = k(x) \cup k(y) \in \mathcal{P}([0,1])$. Then $k(x) \subseteq \pi_1$ and $k(y) \subseteq \pi_1$. Thus, $x, y \in L(k, \pi_1) \neq \emptyset$. By assumption, $L(k, \pi_1)$ is a subalgebra of X and thus $x \cdot y \in L(k, \pi_1)$. So, $k(x \cdot y) \subseteq \pi_1 = k(x) \cup k(y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy subalgebra of X. □

The following theorem can be proved similarly to Theorem 3.12.

Theorem 3.13. *An IHFS $\mathcal{H} = (h, k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is an intuitionistic hesitant fuzzy ideal of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are ideals.*

Definition 3.6. *Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of IHFSs on a reference set X. We define the IHFS $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha = \left(\bigcap_{\alpha\in\Delta} h_\alpha, \bigcup_{\alpha\in\Delta} k_\alpha\right)$ by $\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x) = \bigcap_{\alpha\in\Delta} h_\alpha(x)$ and $\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x) = \bigcup_{\alpha\in\Delta} k_\alpha(x)$ for all $x \in X$, which is called the intuitionistic hesitant intersection of IHFSs.*

Proposition 3.3. *If $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ is a family of intuitionistic hesitant fuzzy ideals of a Hilbert algebra $X = (X, \cdot, 1)$, then $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy ideal of X.*

Proof: Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of intuitionistic hesitant fuzzy ideals of X. Let $x \in X$. Then

$$\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(1) = \bigcap_{\alpha\in\Delta} h_\alpha(1) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(x) = \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x),$$

$$\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(1) = \bigcup_{\alpha\in\Delta} k_\alpha(1) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(x) = \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x).$$

Let $x, y \in X$. Then

$$\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x \cdot y) = \bigcap_{\alpha\in\Delta} h_\alpha(x \cdot y) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(y) = \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(y),$$

$$\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x \cdot y) = \bigcup_{\alpha\in\Delta} k_\alpha(x \cdot y) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(y) = \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(y).$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \bigcap_{\alpha\in\Delta} h_\alpha((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\supseteq \bigcap_{\alpha\in\Delta} (h_\alpha(y_1) \cap h_\alpha(y_2)) \\ &= \left(\bigcap_{\alpha\in\Delta} h_\alpha(y_1)\right) \cap \left(\bigcap_{\alpha\in\Delta} h_\alpha(y_2)\right) \\ &= \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(y_1) \cap \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(y_2),\end{aligned}$$

$$\begin{aligned}\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)((y_1\cdot(y_2\cdot x))\cdot x) &= \bigcup_{\alpha\in\Delta} k_\alpha((y_1\cdot(y_2\cdot x))\cdot x)\\ &\subseteq \bigcup_{\alpha\in\Delta}(k_\alpha(y_1)\cup k_\alpha(y_2))\\ &= \left(\bigcup_{\alpha\in\Delta} k_\alpha(y_1)\right)\cup\left(\bigcup_{\alpha\in\Delta} k_\alpha(y_2)\right)\\ &= \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(y_1)\cup\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(y_2).\end{aligned}$$

Hence, $\bigcap_{\alpha\in\Delta}\mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy ideal of X. □

The following proposition can be proved similarly to Proposition 3.3.

Proposition 3.4. *If $\{\mathcal{H}_\alpha \mid \alpha\in\Delta\}$ is a family of intuitionistic hesitant fuzzy subalgebras of a Hilbert algebra $X=(X,\cdot,1)$, then $\bigcap_{\alpha\in\Delta}\mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy subalgebra of X.*

Definition 3.7. *Let $A=(h_A,k_A)$ and $B=(h_B,k_B)$ be IHFSs on sets X and Y, respectively. The Cartesian product $A\times B=(h,k)$ defined by $h(x,y)=h_A(x)\cap h_B(y)$ and $k(x,y)=k_A(x)\cup k_B(y)$, where $h: X\times Y\to\mathcal{P}([0,1])$ and $k: X\times Y\to\mathcal{P}([0,1])$ for all $x\in X$ and $y\in Y$.*

Remark 3.1. *Let $(X,\cdot,1_X)$ and $(Y,\star,1_Y)$ be Hilbert algebras. Then $(X\times Y,\diamond,(1_X,1_Y))$ is a Hilbert algebra defined by $(x,y)\diamond(u,v)=(x\cdot u,y\star v)$ for every $x,u\in X$ and $y,v\in Y$.*

Proposition 3.5. *If $A=(h_A,k_A)$ and $B=(h_B,k_B)$ are two intuitionistic hesitant fuzzy subalgebras of Hilbert algebras X and Y, respectively, then the Cartesian product $A\times B$ is also an intuitionistic hesitant fuzzy subalgebra of $X\times Y$.*

Proof: Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then

$$\begin{aligned}h((x_1,y_1)\diamond(x_2,y_2)) &= h((x_1\cdot x_2),(y_1\star y_2))\\ &= h_A(x_1\cdot x_2)\cap h_B(y_1\star y_2)\\ &\supseteq (h_A(x_1)\cap h_A(x_2))\cap(h_B(y_1)\cap h_B(y_2))\\ &= (h_A(x_1)\cap h_B(y_1))\cap(h_A(x_2)\cap h_B(y_2))\\ &= h(x_1,y_1)\cap h(x_2,y_2),\\ k((x_1,y_1)\diamond(x_2,y_2)) &= k((x_1\cdot x_2),(y_1\star y_2))\\ &= k_A(x_1\cdot x_2)\cup k_B(y_1\star y_2)\\ &\subseteq (k_A(x_1)\cup k_A(x_2))\cup(k_B(y_1)\cup k_B(y_2))\\ &= (k_A(x_1)\cup k_B(y_1))\cup(k_A(x_2)\cup k_B(y_2))\\ &= k(x_1,y_1)\cup k(x_2,y_2).\end{aligned}$$

Hence, $A\times B$ is an intuitionistic hesitant fuzzy subalgebra of $X\times Y$. □

Theorem 3.14. *Two IHFSs $A=(h_A,k_A)$ and $B=(h_B,k_B)$ are intuitionistic hesitant fuzzy subalgebras of Hilbert algebras X and Y, respectively if and only if the IHFSs $\oplus(A\times B)$ and $\otimes(A\times B)$ are intuitionistic hesitant fuzzy subalgebras of $X\times Y$.*

Proof: It follows from Proposition 3.5 and Theorem 3.4. □

The following proposition can be proved similarly to Proposition 3.5.

Proposition 3.6. *If $A=(h_A,k_A)$ and $B=(h_B,k_B)$ are two intuitionistic hesitant fuzzy ideals of Hilbert algebras X and Y, respectively, then the Cartesian product $A\times B$ is also an intuitionistic hesitant fuzzy ideal of $X\times Y$.*

Theorem 3.15. *Two IHFSs $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy ideals of Hilbert algebras X and Y, respectively if and only if the IHFSs $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy ideals of $X \times Y$.*

Proof: It follows from Proposition 3.6 and Theorem 3.6. □

A mapping $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1_X) = 1_Y$.

Definition 3.8. *Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{H} = (h, k)$ is an IHFS on Y, then the IHFS $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ in X is called the pre-image of $\mathcal{H}$ under f.*

Theorem 3.16. *Let $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ be a homomorphism of Hilbert algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy ideal of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy ideal of X.*

Proof: By assumption, $h(f(1_X)) = h(1_X) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(1_X) = h(f(1_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(1_X)) = k(1_Y) \subseteq k(y)$ for every $y \in Y$. In particular, $(k \circ f)(1_X) = k(f(1_X)) \subseteq k(f(x)) = (k \circ f)(x)$ for all $x \in X$. Let $x, y \in X$. Then, by the assumption,

$$(h \circ f)(x \cdot y) = h(f(x \cdot y)) = h(f(x) \star f(y)) \supseteq h(f(y)) = (h \circ f)(y),$$
$$(k \circ f)(x \cdot y) = k(f(x \cdot y)) = k(f(x) \star f(y)) \subseteq k(f(y)) = (k \circ f)(y).$$

Let $x, y_1, y_2 \in X$. Then, by assumption,

$$\begin{aligned}
(h \circ f)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= h(f((y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&= h((f(y_1) \star (f(y_2) \star f(x))) \star f(x)) \\
&\supseteq h(f(y_1)) \cap h(f(y_2)) \\
&= (h \circ f)(y_1) \cap (h \circ f)(y_2), \\
(k \circ f)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= k(f((y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&= k((f(y_1) \star (f(y_2) \star f(x))) \star f(x)) \\
&\subseteq k(f(y_1)) \cup k(f(y_2)) \\
&= (k \circ f)(y_1) \cup (k \circ f)(y_2).
\end{aligned}$$

Hence, $f^{-1}(\mathcal{H})$ is an intuitionistic hesitant fuzzy ideal of X. □

The following theorem can be proved similarly to Theorem 3.16.

Theorem 3.17. *Let $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ be a homomorphism of Hilbert algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy subalgebra of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy subalgebra of X.*

4. **Conclusion.**

In the present paper, we have introduced the concepts of intuitionistic hesitant fuzzy subalgebras and ideals of Hilbert algebras. The relationship between intuitionistic hesitant fuzzy subalgebras (ideals) and their π-level subsets is described. Moreover, the homomorphic pre-images of intuitionistic hesitant fuzzy subalgebras (ideals) of Hilbert algebras are also studied and some related properties are investigated.

To extend the results of this paper, future research will focus on intuitionistic hesitant fuzzy sets in the concept of anti-type in Hilbert algebras. It can also be applied to other algebraic systems, and the results can be compared to those presented in this article.

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Author Biography

Aiyared Iampan is an Associate Professor at Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Naresuan University, Phitsanulok, Thailand, under the thesis advisor of Professor Dr. Manoj Siripitukdet. His areas of interest include algebraic theory of semigroups, ternary semigroups, and Γ-semigroups, lattices and ordered algebraic structures, fuzzy algebraic structures, and logical algebras. He was the founder of the Group for Young Algebraists in University of Phayao in 2012 and one of the co-founders of the Fuzzy Algebras and Decision-Making Problems Research Unit in University of Phayao in 2021.

Ramasamy Subasini works at Department of Mathematics, Pollachi Institute of Engineering and Technology, Tamilnadu, India. She received her B.Sc., M.Sc., and M.Phil. degrees from N.G.M. College, Bharathiar University, Coimbatore, India. She has published 6 research papers in various international journals. Her area of interest includes topology and fuzzy algebra.

Ponnusamy Maragatha Meenakshi is working as an Associate Professor of Mathematics in Thanthai Periyar Government Arts and Science College, affiliated to Bharathidasan University, Tamilnadu, India. She received her B.Sc. and M.Sc. degrees in Mathematics from Bharathiyar University, Coimbatore. She has completed her computer diploma PBDCSA and M.Phil. in Mathematics from Madurai Kamaraj University, and received her Ph.D. in Mathematics from Bharathidasan University, Tiruchirappalli. She has passed CSIR-UGC NET exam in June 1997. Her areas of research are computational fluid dynamics and topology. She has published around 20 papers in various international journals. Some of her papers are presented at the International Conferences BAIL. Five students had completed the Ph.D. degrees and seven are currently working under her supervision.

Neelamegarajan Rajesh works at Department of Mathematics, Rajah Serfoji Government College, affliated to Bharathidasan University, Tamilnadu, India. He received his B.Sc., M.Sc., and M.Phil. degrees in Mathematics from Bharathidasan University, Trichy, India and Ph.D. degree from Madurai Kamaraj University, India. He has published more than 350 research papers in various international journals. His areas of interest include topology and fuzzy algebra.