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NEW APPROACH TOWARDS DIFFERENT TYPES OF BI-QUASI IDEALS IN b-SEMIRINGS AND ITS EXTENSION

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Abstract. *We introduce two types of bi-quasi ideals in b-semirings. Each bi-quasi ideal generated by a single element and set is established. For any $a \in S$, the generalized 1-right bi-quasi ideal generated by "a" is $< a >_{g1rbq} = \{a\} \cup [(a\diamondsuit_2 S) \cap (a\diamondsuit_2 S\diamondsuit_2 a)]$. We characterize various 1-regular (2-regular) by using generalized 1-bi-quasi ideal, 1-bi-quasi ideal, weak-1-right ideal, weak-1-left ideal, right ideal, left ideal, bi-ideal, quasi-ideal (generalized 2-bi-quasi ideal, 2-bi-quasi ideal, weak-2-right ideal, weak-2-left ideal, right ideal, left ideal, bi-ideal, quasi-ideal). Every quasi-ideal is a bi-quasi ideal, and the reverse implication does not hold. Examples are provided to strengthen our results. The motivation of this paper is to show that the categories of quasi-ideals over b-semirings have gained prominence in algebraic structures. Although many other algebraic concepts of these valuations are worthy of a more extensive study, here we only settle down the algebraic considerations that we will need in order to investigate their 1-bi-quasi ideals and 2-bi-quasi ideals in b-semirings.*

Keywords: Generalized 1-bi-quasi ideal, 1-bi-quasi ideal, Generalized 2-bi-quasi ideal, 2-bi-quasi ideal

1. **Introduction.** Numerous studies have described various forms of ideals in algebraic structures like semirings [1] and rings [2]. Vandiver inaugurated the proposal of semirings as a part of the generalization of rings [3]. In general, ideal theory for semirings need not be consistent with commutative properties under any operation. Several authors have studied aspects of algebraic structures such as semigroups, semirings and rings. In 1952, Good and Hughes [4] presented the idea of bi-ideals for semigroups. Additionally, it is a specific instance of (m,n)-ideal introduced by Lajos and Szász [5]. The origin of bi-ideals based on associative rings was introduced by Lajos and Szász [5]. Quasi-ideals are generalizations of both left and right ideals. The concept of quasi-ideals was introduced

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by Steinfeld [6] for semigroups and rings in 1956. In fact, the notion of one-sided ideals of rings and semigroups can be regarded as a generalization of the notion of ideals of rings and semigroups. In general, semigroups are generalizations of semirings and groups.

Von Neumann [24] defined a ring to be a regular ring. The concept of b-semirings [7] was introduced by Chinram in 2009. The algebraic structure $(S, \diamondsuit_1, \diamondsuit_2)$ is called a b-semiring if $(S, \diamondsuit_1)$ and $(S, \diamondsuit_2)$ are semigroups, connected by four distributive laws that "$\diamondsuit_2$" distributes over "$\diamondsuit_1$" from left and right and "$\diamondsuit_1$" distributes over "$\diamondsuit_2$" from left and right [7]. The subset A of S is called a sub b-semiring in S if A is itself a b-semiring [7]. The concept of weak-1(2)-right ideal, weak-1(2)-left ideal, weak-1(2)-ideal in b-semirings are introduced by Chinram [7]. Recently, Mohanraj and Palanikumar [8] introduced that 1-regular (2-regular, regular) b-semirings are characterized by using various weak-ideals. Palanikumar et al. [9] discussed different prime partial bi-ideals in noncommutative partial rings. Palanikumar et al. [10] interacted M-tri-basis of an ordered Γ-semigroup. Recently, Palanikumar et al. discussed some algebraic structures such as semirings, and ring semigroups [11, 12, 13, 14, 15, 16, 17]. Palanikumar and Mohanraj [18] discussed the different types of quasi-ideals in b-semirings.

There are two recent applications of special b-semirings; see [19]. An innovative concept is introduced in the modelling operations of container quay cranes based on pseudo-analysis in the form of max-plus semiring. The b-semiring theory is applied in many fields, e.g., geometry, topology, differential equations, in the automata theory. There are some differences in defining a b-semiring, and there are different terms used for special b-semirings, e.g., path algebras, dioids, max-plus algebras, min-max algebras, max algebras, and min-plus algebras. On a b-semiring, the ordering can be partial or full. It is also possible to form matrices over a b-semiring on different domains, with specially defined operations pointing out that square matrices over a b-semiring also form a b-semiring. We present a model of container quay cranes with several reloading places (three and six), simulation models and results.

We investigate the ordering on a set of monotone doubly stochastic matrices by forming a b-semiring in which an ordering induces mobility measures in the Shorrocks' sense. It is also applied in b-semirings [20]. It is a fairly natural generalization of fuzzy set theory, all the objects of the universe are characterized by their membership and non-membership degrees, and their sum is always bounded by one [21]. It is a parameterization tool for dealing with uncertainty. Softsets, when compared to the run certain theories, more closely reflect the objectivity and complexity of decision-making in real-world circumstances [22]. Data were collected related to attributes of the college education to demonstrate the significance of decision-making in the case of ten colleges [23]. A study also introduces neutrosophic set into the evaluation of sustainable financing policies aimed at reducing environmental pollution.

This paper discusses several important classical results in 1-bi-quasi ideals and is characterized by 1-quasi ideals and various weak ideals. This paper extends the notion of various quasi-ideals into various bi-quasi ideals. The paper is organized into five sections as follows. Section 1 is referred to as an introduction. There is a brief description of b-semirings in Section 2 as well as relevant definitions and results. Section 3 discusses the 1-bi-quasi ideal generated by a single element and subset with numerical examples. The central result in this section is Theorem 3.1, which gives a complete characterization of bi-quasi ideals in b-semirings. Section 4 discusses the 2-bi-quasi ideal generated by a single element and subset with numerical examples, and Theorem 4.7 was given to improve the characterization in this case. Finally, the future scope of our results is from the perspective of hyper b-semirings using bi-ideals and bi-quasi ideals conclusion is provided in Section 5.

The objective of this paper:

1) To define the various bi-quasi ideals in b-semirings.
2) We are going to demonstrate the generator of single elements and subset for bi-quasi ideals in b-semirings.
3) The intersection of a weak-1-right ideal (weak-1-left ideal) and weak-1 bi-ideal in S is neither weak-1-right ideal (weak-1-left ideal) nor weak-1 bi-ideal by Example 3.3.
4) What is the intersection of weak-1-right ideal (weak-1-left ideal) with weak-1 bi-ideal? We answer the questions by introducing 1-bi-quasi ideal.

2. Preliminaries.

Definition 2.1. *Let A and B be the subsets of $(S, \diamondsuit_1, \diamondsuit_2)$. Then the $\diamondsuit_1$product and $\diamondsuit_2$product of A and B, denoted by $A\diamondsuit_1 B$ and $A\diamondsuit_2 B$ respectively are defined as follows: $A\diamondsuit_1 B = \{a\diamondsuit_1 b | a \in A \text{ and } b \in B\}$ and $A\diamondsuit_2 B = \{a\diamondsuit_2 b | a \in A \text{ and } b \in B\}$.*

Definition 2.2. *The subset A of S is called a weak-1-right ideal (weak-1-left ideal) of S if $a_1\diamondsuit_1 a_2 \in A$ and $a_1\diamondsuit_2 s \in A$ ($s\diamondsuit_2 a_1 \in A$) for all $a_1, a_2 \in A$ and $s \in S$.*

Definition 2.3. *The subset A of S is called a weak-1 ideal of S if it is both weak-1-right ideal and weak-1-left ideal of S.*

Definition 2.4. *The subset A of S is called a weak-2-right ideal of S (weak-2-left ideal) if $a_1\diamondsuit_2 a_2 \in A$ and $a_1\diamondsuit_1 s \in A$ ($s\diamondsuit_1 a_1 \in A$) for all $a_1, a_2 \in A$ and $s \in S$.*

Definition 2.5. *The subset A of S is called a weak-2 ideal of S if it is both a weak-2-right ideal and a weak-2-left ideal of S.*

Definition 2.6. *The subset A of S is called a right (left) ideal of S if it is both a weak-1-right (left) ideal and a weak-2-right (left) ideal of S.*

Definition 2.7. *(i) The subset Q of S is called a generalized 1-quasi ideal in S if $(Q\diamondsuit_2 S) \cap (S\diamondsuit_2 Q) \subseteq Q$.*
(ii) The generalized 1-quasi ideal Q is called a 1-quasi ideal in S if Q is a sub b-semiring.

Definition 2.8. *(i) The subset Q of S is called a generalized 2-quasi ideal in S if $(Q\diamondsuit_1 S) \cap (S\diamondsuit_1 Q) \subseteq Q$.*
(ii) The generalized 2-quasi ideal Q is called a 2-quasi ideal in S if Q is a sub b-semiring.

Definition 2.9. [8] *(i) The b-semiring S is called 1-regular [2-regular] if for each $a \in S$ there exists $x \in S$ such that $a\diamondsuit_2(x\diamondsuit_2 a) = a$ $[a\diamondsuit_1(x\diamondsuit_1 a) = a]$.*
(ii) The b-semiring S is called regular if it is both 1-regular and 2-regular in S.

3. Type-1 Bi-Quasi Ideals of b-Semirings.

In this section, $\diamondsuit_1$ and $\diamondsuit_2$ represent min-max-product and max-min-product, respectively.

The intersection of a weak-1-right ideal (weak-1-left ideal) and weak-1 bi-ideal in S is neither weak-1-right (left) ideal nor weak-1 bi-ideal in S by the following Example 3.3.

What is the intersection of weak-1-right (left) ideal with weak-1 bi-ideal? We answer the questions by introducing the 1-bi-quasi ideal.

Notations: For a subset A of S and $i = 1, 2, 3, \ldots, n$,

(i) $\sum A = \{(a_1\diamondsuit_1 a_2\diamondsuit_1 \cdots \diamondsuit_1 a_n) | a_i \in A\}$.

(ii) $\prod A = \{(a_1\diamondsuit_2 a_2\diamondsuit_2 \cdots \diamondsuit_2 a_n) | a_i \in A\}$.

(iii) $\sum(A\diamondsuit_2 S) = \{(a_1\diamondsuit_2 s_1)\diamondsuit_1(a_2\diamondsuit_2 s_2)\diamondsuit_1 \cdots \diamondsuit_1(a_n\diamondsuit_2 s_n) | a_i \in A,\ s_i \in S\}$.

(iv) $\prod(A\diamondsuit_1 S) = \{(a_1\diamondsuit_1 s_1)\diamondsuit_2(a_2\diamondsuit_1 s_2)\diamondsuit_2 \cdots \diamondsuit_2(a_n\diamondsuit_1 s_n) | a_i \in A,\ s_i \in S\}$.

(v) $\sum(A\diamondsuit_2 S\diamondsuit_2 A) = \{(a_1\diamondsuit_2 s_1\diamondsuit_2 a_1)\diamondsuit_1(a_2\diamondsuit_2 s_2\diamondsuit_2 a_2)\cdots\diamondsuit_1(a_n\diamondsuit_2 s_n\diamondsuit_2 a_n)|a_i \in A, s_i \in S\}$.

(vi) $\prod(A\diamondsuit_1 S\diamondsuit_1 A) = \{(a_1\diamondsuit_1 s_1\diamondsuit_1 a_1)\diamondsuit_2(a_2\diamondsuit_1 s_2\diamondsuit_1 a_2)\cdots\diamondsuit_2(a_n\diamondsuit_1 s_n\diamondsuit_1 a_n)|a_i\in A, s_i \in S\}$.

Definition 3.1. *(i) The subset Q of S is called a generalized 1-right bi-quasi ideal in S if $(Q\diamondsuit_2 S) \cap (Q\diamondsuit_2 S\diamondsuit_2 Q) \subseteq Q$.*
(ii) The generalized 1-right bi-quasi ideal Q is called a 1-right bi-quasi ideal in S if Q is a sub b-semiring.

Definition 3.2. *(i) The subset Q of S is called a generalized 1-left bi-quasi ideal in S if $(Q\diamondsuit_2 S\diamondsuit_2 Q) \cap (S\diamondsuit_2 Q) \subseteq Q$.*
(ii) The generalized 1-left bi-quasi ideal Q is called a 1-left bi-quasi ideal in S if Q is a sub b-semiring.

Lemma 3.1. *The generalized 1-right bi-quasi ideal Q is a 1-right bi-quasi ideal in S if Q is closed under "$\diamondsuit_1$".*

Proof: Suppose that Q is a generalized 1-right bi-quasi ideal and its closed under "$\diamondsuit_1$". Thus, Q is a 1-right bi-quasi ideal in S. □

Remark 3.1. *Every 1-right (left) bi-quasi ideal is a generalized 1-right (left) bi-quasi ideal. The Converse of the Remark 3.1 is not true by Example 3.1.*

Example 3.1. *Consider $(S, \diamondsuit_1, \diamondsuit_2)$ the b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\}.$$

$$Let\ Q = \left\{ \begin{pmatrix} 0 & q_1 & 0 & 0 \\ 0 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| q_i^{'s} \in Z^* \right\}.$$

$$Then\ (Q *_2 S *_2 Q) \cap (S *_2 Q) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \subseteq Q.$$

Hence, Q is a 1-generalized left bi-quasi ideal but not a 1-left bi-quasi ideal of S.

Theorem 3.1. *Every 1-quasi ideal is 1-bi-quasi ideal in S.*

Proof: Suppose that Q is a 1-quasi ideal of S. Now, $(Q\diamondsuit_2 S) \cap (Q\diamondsuit_2 S\diamondsuit_2 Q) \subseteq Q$ and $(S\diamondsuit_2 Q) \cap (Q\diamondsuit_2 S\diamondsuit_2 Q) \subseteq Q$. □

Example 3.2. *Consider $(S, \diamondsuit_1, \diamondsuit_2)$ the b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 & 0 \\ s_4 & s_5 & s_6 & 0 & 0 & 0 \\ s_7 & s_8 & s_9 & s_{10} & 0 & 0 \\ s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\}.$$

$$\text{Let } Q = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| q_i^{'s} \in Z^* \right\}.$$

Hence, Q is a 1-bi-quasi ideal but not a 1-quasi ideal of S by

$$(Q *_2 S) \cap (S *_2 Q) = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ o_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| o_i^{'s} \in Z^* \right\}.$$

Remark 3.2. *1) Every generalized 1-quasi ideal is a generalized 1-bi-quasi ideal in S.*
2) Every weak-1-right (left) ideal is a 1-right (left) bi-quasi ideal.

Theorem 3.2. *The intersection of a weak-1-right (left) ideal and a weak-1 bi-ideal is a 1-right (left) bi-quasi ideal.*

Proof: For the weak-1-right (left) ideal A and weak-1 bi-ideal B in S, $A \cap B$ is a sub b-semiring. Thus, $A \cap B$ is a 1-right bi-quasi ideal in S. □

Example 3.3. *Consider $(S, \diamondsuit_1, \diamondsuit_2)$ the b-semiring.*

$$\text{Let } S = \left\{ \left. \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| s_i^{'s} \in Z^* \right\}.$$

$$\text{Then } \{(A \cap B)\diamondsuit_2 S \diamondsuit_2 (A \cap B)\} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| \in Z^* \right\} \subseteq Q.$$

Hence, $A \cap B$ is a 1-right bi-quasi ideal, but neither right ideal A nor bi-ideal B of S.

Theorem 3.3. *For any $a \in S$, the generalized 1-right bi-quasi ideal generated by "a" is $< a >_{g1rbq} = \{a\} \cup [(a\diamondsuit_2 S) \cap (a\diamondsuit_2 S\diamondsuit_2 a)]$.*

Corollary 3.1. *For a subset A of S, $A \cup [\sum(A\diamondsuit_2 S) \cap \sum(A\diamondsuit_2 S\diamondsuit_2 A)]$ is the generalized 1-right bi-quasi ideal generated by a set A in S.*

Proof: Let $x \in \{A\} \cup [\sum(A\diamondsuit_2 S) \cap \sum(A *_2 S\diamondsuit_2 A)]$. If B is a generalized 1-right bi-quasi ideal in S such that $A \subseteq B$, then $\{A\} \cup [\sum(A\diamondsuit_2 S) \cap \sum(A\diamondsuit_2 S\diamondsuit_2 A)] \subseteq B$. Thus, $< A >_{g1rbq}$ is the generalized 1-right bi-quasi ideal generated by A. □

Corollary 3.2. *For a subset A of S, $A \cup [\sum(S\diamondsuit_2 A) \cap \sum(A\diamondsuit_2 S\diamondsuit_2 A)]$ is the generalized 1-left bi-quasi ideal generated by a set A in S.*

Theorem 3.4. *For any $a \in S$, the 1-right bi-quasi ideal generated by "a", denoted by $< a >_{1rbq}$ is given by $\{na | n \in \mathbb{Z}^+\} \cup [(a\diamondsuit_2 S) \cap (a\diamondsuit_2 S\diamondsuit_2 a)]$.*

Theorem 3.5. *For any $a \in S$, the 1-left bi-quasi ideal generated by "a", denoted by $< a >_{1lbq}$ is given by $\{na | n \in \mathbb{Z}^+\} \cup [(S\diamondsuit_2 a) \cap (a\diamondsuit_2 S\diamondsuit_2 a)]$.*

Corollary 3.3. *For the subset A of S, $\sum A \cup [\sum(A\diamondsuit_2 S) \cap \sum(S\diamondsuit_2 A)]$ is the 1-right bi-quasi ideal generated by a set A in S.*

Theorem 3.6. *For a b-semiring S, the following statements are equivalent.*
(1) S is 1-regular.
(2) $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$, for the weak-1-right ideals R and generalized 1-right bi-quasi ideals Q_1.
(3) $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$, for the weak-1-right ideals R and 1-right bi-quasi ideals Q_1.
(4) $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$, for the weak-1-right ideals R and generalized 1-left bi-quasi ideals Q_1.
(5) $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$, for the weak-1-right ideals R and 1-left bi-quasi ideals Q_1.
(6) $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$, for the weak-1-right ideals R and generalized 1-quasi ideals Q_1.
(7) $R \cap Q \subseteq R\diamondsuit_2 Q$, for the weak-1-right ideals R and 1-quasi ideals Q.
(8) $Q_1 \cap L \subseteq Q_1\diamondsuit_2 L$, for the generalized 1-right bi-quasi ideals Q_1 and weak-1-left ideals L.
(9) $Q_1 \cap L \subseteq Q_1\diamondsuit_2 L$, for the generalized 1-left bi-quasi ideals Q_1 and weak-1-left ideals L.
(10) $Q_1 \cap L \subseteq Q_1\diamondsuit_2 L$, for the generalized 1-quasi ideals Q_1 and weak-1-left ideals L.
(11) $Q \cap L \subseteq Q\diamondsuit_2 L$, for the 1-quasi ideals Q and weak-1-left ideals L.
(12) $R \cap L = R\diamondsuit_2 L$, for the weak-1-right ideals R and weak-1-left ideals L.

Proof: First, we prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (12), (1) $\Rightarrow$ (8) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$ (1), (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (7) and (2) $\Rightarrow$ (6) $\Rightarrow$ (12), (1) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (12) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) For $a \in R \cap Q_1$, then there exists $s \in S$ such that $a = (a\diamondsuit_2 s)\diamondsuit_2 a$. Thus, $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$.
(2) $\Rightarrow$ (3) By Remark 3.1, the result holds.
(3) $\Rightarrow$ (7) By Theorem 3.1, (7) follows.
(7) $\Rightarrow$ (12) By Remark 3.2, $R \cap L \subseteq R\diamondsuit_2 L$. Now, $R\diamondsuit_2 L \subseteq R\diamondsuit_2 S \subseteq R$ and $R\diamondsuit_2 L \subseteq S\diamondsuit_2 L \subseteq L$. Therefore, $R\diamondsuit_2 L \subseteq R \cap L$. Hence, $R\diamondsuit_2 L = R \cap L$. Then (12) follows.
(1) $\Rightarrow$ (8) For $a \in Q_1 \cap L$, then there exists $s \in S$ such that $a = a\diamondsuit_2(s\diamondsuit_2 a)$. Thus, $Q_1 \cap L \subseteq Q_1\diamondsuit_2 L$.
(8) $\Rightarrow$ (10) The proof follows from Remark 3.2.
(10) $\Rightarrow$ (11) The proof follows from Lemma 3.3 [18], and the result holds.
(11) $\Rightarrow$ (12) The proof follows from Theorem 3.9 [18].
(1) $\Rightarrow$ (4) For $a \in R \cap Q_1$, then there exists $s \in S$ such that $a = (a\diamondsuit_2 s)\diamondsuit_2 a$. Thus, $R \cap Q_1 \subseteq R\diamondsuit_2 Q_1$.
(4) $\Rightarrow$ (5) By Theorem 3.1, then (5) follows.
(5) $\Rightarrow$ (7) The proof follows from Theorem 3.1.
(2) $\Rightarrow$ (6) By Theorem 3.1, the proof follows.
(6) $\Rightarrow$ (12) The proof follows from Theorem 3.9 [18], and the result holds.
(1) $\Rightarrow$ (9) For $a \in Q_1 \cap L$, then there exists $s \in S$ such that $a = a\diamondsuit_2(s\diamondsuit_2 a)$. Thus, $Q_1 \cap L \subseteq Q_1\diamondsuit_2 L$.
(9) $\Rightarrow$ (10) By Theorem 3.1, the proof follows.
(10) $\Rightarrow$ (12) By Remark 3.2, then (12) holds.
(12) $\Rightarrow$ (1) The proof follows from Theorem 3.16 [18]. □

Theorem 3.7. *For a b-semiring S, the following statements are equivalent.*
(1) S is 1-regular.
(2) $B_1 \cap I \cap B_2 \subseteq B_1\diamondsuit_2 I\diamondsuit_2 B_2$, for the generalized bi-quasi ideals B_1 and B_2 and weak-1 ideals I.
(3) $B \cap I \cap Q \subseteq B\diamondsuit_2 I\diamondsuit_2 Q$, for the generalized bi-quasi ideals B, weak-1 ideals I and

1-quasi ideals Q.
(4) $Q \cap I \cap B \subseteq Q\diamondsuit_2 I\diamondsuit_2 B$, for the 1-quasi ideals Q, weak-1 ideals I and generalized bi-quasi ideals B.
(5) $B \cap I \cap B \subseteq B\diamondsuit_2 I\diamondsuit_2 B$, for the bi-quasi ideals B and weak-1 ideals I.
(6) $B_1 \cap I \cap L \subseteq B_1\diamondsuit_2 I\diamondsuit_2 L$, for the generalized 1-bi-quasi ideals B_1, weak-1 ideals I and weak-1-left ideals L.
(7) $B \cap I \cap L \subseteq B\diamondsuit_2 I\diamondsuit_2 L$, for the bi-quasi ideals B, weak-1 ideals I and weak-1-left ideals L.
(8) $R \cap I \cap B_2 \subseteq R\diamondsuit_2 I\diamondsuit_2 B_2$, for the weak-1-right ideals R, weak-1 ideals I and generalized bi-quasi ideals B_2.
(9) $R \cap I \cap B \subseteq R\diamondsuit_2 I\diamondsuit_2 B$, for the weak-1-right ideals R, weak-1 ideals I and bi-quasi ideals B.
(10) $R \cap I \cap L \subseteq R\diamondsuit_2 I\diamondsuit_2 L$, for the weak-1-right ideals R, weak-1 ideals I and weak-1-left ideals L.
(11) $R \cap L = R\diamondsuit_2 L$, for the weak-1-right ideals R and weak-1-left ideals L.
(12) $B_1 \cap I \subseteq B_1\diamondsuit_2 I\diamondsuit_2 B_1$, for the generalized bi-quasi ideals B_1 and weak-1 ideals I.
(13) $B \cap I \subseteq B\diamondsuit_2 I\diamondsuit_2 B$, for the bi-quasi ideals B and weak-1 ideals I.
(14) $B_1 = B_1\diamondsuit_2 S\diamondsuit_2 B_1$, for the generalized bi-quasi ideals B_1.
(15) $Q = Q\diamondsuit_2 S\diamondsuit_2 Q$, for the 1-quasi ideals Q.

Proof: First, we prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (1), (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (2) $\Rightarrow$ (5) $\Rightarrow$ (9) $\Rightarrow$ (15) $\Rightarrow$ (1), (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (10), (2) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (15) $\Rightarrow$ (1) and (12) $\Rightarrow$ (14) $\Rightarrow$ (15) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) For $a \in Q_1 \cap I \cap Q_2$, then there exists $s \in S$ such that $a = a\diamondsuit_2 s\diamondsuit_2 a$. Thus, $a = a\diamondsuit_2(s\diamondsuit_2 a\diamondsuit_2 s)\diamondsuit_2 a \in Q_1\diamondsuit_2 I\diamondsuit_2 Q_2$. Thus (2) holds.
(2) $\Rightarrow$ (3) Straightforward.
(3) $\Rightarrow$ (7) The proof follows from Theorem 3.9 [18].
(7) $\Rightarrow$ (10) By Remark 3.2, then (10) holds.
(10) $\Rightarrow$ (11) Taking $I = S$ in (5), $R \cap L \subseteq R\diamondsuit_2 L$. For weak-1-right ideal R and weak-1-left ideal L, $R\diamondsuit_2 L \subseteq R\diamondsuit_2 S \subseteq R$ and $R\diamondsuit_2 L \subseteq S\diamondsuit_2 L \subseteq L$. Therefore, $R\diamondsuit_2 L \subseteq R \cap L$. Thus, $R \cap L = R\diamondsuit_2 L$.
(11) $\Rightarrow$ (1) The proof follows from Theorem 3.16 [18].
(2) $\Rightarrow$ (4) Straightforward.
(4) $\Rightarrow$ (8) The proof follows from Theorem 3.9 [18], and we get the result.
(8) $\Rightarrow$ (9) Straightforward.
(9) $\Rightarrow$ (10) By Remark 3.2, then (10) holds.
(2) $\Rightarrow$ (5) Straightforward.
(5) $\Rightarrow$ (9) By Remark 3.2, the proof follows.
(9) $\Rightarrow$ (15) Straightforward.
(1) $\Rightarrow$ (6) For $a \in B_1 \cap I \cap L$, then there exists $s \in S$ such that $a = a\diamondsuit_2 s\diamondsuit_2 a$. Thus, $a = a\diamondsuit_2(s\diamondsuit_2 a\diamondsuit_2 s)\diamondsuit_2 a \in B_1\diamondsuit_2 I\diamondsuit_2 L$. Thus (6) holds.
(6) $\Rightarrow$ (7) Straightforward.
(2) $\Rightarrow$ (12) Taking $B_2 = B_1$ in (2), we get the result.
(12) $\Rightarrow$ (13) Straightforward.
(13) $\Rightarrow$ (15) Straightforward.
(12) $\Rightarrow$ (14) Taking $I = S$, $B_1 \subseteq B_1\diamondsuit_2 S\diamondsuit_2 B_1 \subseteq [(B_1\diamondsuit_2 S) \cap (S\diamondsuit_2 B_1)] \subseteq B_1$ implies $B_1 = B_1\diamondsuit_2 S\diamondsuit_2 B_1$.
(14) $\Rightarrow$ (15) Straightforward.
(15) $\Rightarrow$ (1) For any $a \in S$ by (15), $a \in < a >_{1q} \diamondsuit_2 S\diamondsuit_2 < a >_{1q}$ and by Theorem 3.5 and Lemma 3.1. Thus, $a \in a\diamondsuit_2 S\diamondsuit_2 a$. Hence, S is 1-regular. $\square$

In the above, we have discussed the generalized bi-quasi ideals and the equivalence statement was proved under the operation max-min product $\diamondsuit_2$ in type-1 bi-quasi ideals in b-semirings whereas the similar condition is also satisfying by the operator min-max-product $\diamondsuit_1$ in type-2 bi-quasi ideals in b-semirings.

4. **Type-2-Bi-Quasi Ideals in *b*-Semirings.** The intersection of a weak-2-right ideal (weak-2-left ideal) and weak-2 bi-ideal in S is neither weak-2-right (left) ideal nor weak-2 bi-ideal in S.

What is the intersection of weak-2-right (left) ideal with weak-2 bi-ideal? We answer the questions by introducing the 2-bi-quasi ideal.

Lemma 4.1. *The generalized 2-right bi-quasi ideal Q is a 2-right bi-quasi ideal in S if Q is closed under "$\diamondsuit_2$".*

Proof: Suppose that Q is a generalized 2-right bi-quasi ideal and its closed under "$\diamondsuit_2$". Thus, Q is a 2-right bi-quasi ideal in S. □

Remark 4.1. *1) Every 2-right (left) bi-quasi ideal is a generalized (left) 2-right bi-quasi ideal.*
2) Converse of Remark 4.1 is not true by Example 4.1.

Example 4.1. *Consider $(S, \diamondsuit_2, \diamondsuit_1)$ the b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & 0 & 0 \\ s_7 & s_8 & s_9 & s_{10} \\ s_{11} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\}$$

$$Let\ Q = \left\{ \begin{pmatrix} q_1 & q_2 & 0 & 0 \\ 0 & q_3 & 0 & 0 \\ q_4 & 0 & q_5 & q_6 \\ q_7 & 0 & 0 & 0 \end{pmatrix} \middle| q_i^{'s} \in Z^* \right\}$$

$$Then\ (Q *_1 S *_1 Q) \cap (S *_1 Q) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| p_i^{'s} \in Z^* \right\} \subseteq Q.$$

Hence, Q is a generalized 2-left bi-quasi ideal but not a 2-left bi-quasi ideal of S.

Theorem 4.1. *Every 2-quasi ideal is 2-bi-quasi ideal in S.*

Proof: Suppose that Q is a 2-quasi ideal of S. Now, $(Q\diamondsuit_1 S) \cap (Q\diamondsuit_1 S\diamondsuit_1 Q) \subseteq (Q\diamondsuit_1 S) \cap (S\diamondsuit_1 Q) \subseteq Q$ and $(S\diamondsuit_1 Q) \cap (Q\diamondsuit_1 S\diamondsuit_1 Q) \subseteq (S\diamondsuit_1 Q) \cap (Q\diamondsuit_1 S) \subseteq Q$. □

Example 4.2. *Consider $(S, \diamondsuit_2, \diamondsuit_1)$ the b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & 0 & 0 \\ s_7 & s_8 & s_9 & s_{10} \\ s_{11} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\}$$

$$Let\ Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_3 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \middle| q_i^{'s} \in Z^* \right\}$$

$$Then\ (Q *_1 S) \cap (Q *_1 S *_1 Q) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \subseteq Q.$$

Hence, Q is a 2-bi-quasi ideal but not a 2-quasi ideal of S.

Remark 4.2. *1) Every generalized 2-quasi ideal is a generalized 2-bi-quasi ideal in S. 2) Every weak-2-right (left) ideal is a 2-right (left) bi-quasi ideal.*

Theorem 4.2. *The intersection of a weak-2-right (left) ideal and a weak-2 bi-ideal is a 2-right (left) bi-quasi ideal.*

Proof: For the weak-2-right (left) ideal A and weak-2 bi-ideal B in S, $A \cap B$ is a sub b-semiring. Thus, $A \cap B$ is a 2-right bi-quasi ideal in S. □

Theorem 4.3. *For any $a \in S$, the generalized 2-right bi-quasi ideal generated by "a" is $< a >_{g2rbq} = \{a\} \cup [(a\diamondsuit_1 S) \cap (a\diamondsuit_1 S\diamondsuit_1 a)]$.*

Corollary 4.1. *For a subset A of S, $A \cup [\prod(A\diamondsuit_1 S) \cap \prod(A\diamondsuit_1 S\diamondsuit_1 A)]$ is the generalized 2-right bi-quasi ideal generated by a set A in S.*

Proof: Clearly, $\prod A\ \cup [\prod(A\diamondsuit_1 S) \cap \prod(A *_1 S\diamondsuit_1 A)]$ is the generalized 2-right bi-quasi ideal and $\prod A$ is closed under $\diamondsuit_2$. For $x, y \in \prod(A\diamondsuit_1 S) \cap \prod(A *_1 S\diamondsuit_1 A)$. Then $x = (a_1\diamondsuit_1 s_1)\diamondsuit_2(a_2\diamondsuit_1 s_2)\diamondsuit_2 \cdots \diamondsuit_2(a_n\diamondsuit_1 s_n)$, $x = (a_1'\diamondsuit_1 s_1'\diamondsuit_1 a_1'')\diamondsuit_2 \cdots \diamondsuit_2(a_n'\diamondsuit_1 s_n'\diamondsuit_1 a_n'')$ and $y = (a_1'''\diamondsuit_1 s_1'')\diamondsuit_2(a_2'''\diamondsuit_1 s_2'')\diamondsuit_2 \cdots \diamondsuit_2(a_n'''\diamondsuit_1 s_n'')$. Hence, $\prod A \cup [\prod(A\diamondsuit_1 S) \cap \prod(A\diamondsuit_1 S\ \diamondsuit_1 A)]$ is the 2-quasi ideal in S. If B is a 2-quasi ideal in S such that $A \subseteq B$, then $\prod A \cup [\prod(A\diamondsuit_1 S) \cap \prod(S\diamondsuit_1 A)] \subseteq B$. Thus, $< A >_{1q}$ is the 2-quasi ideal generated by A. □

Corollary 4.2. *For a subset A of S, $A \cup [\prod(S\diamondsuit_1 A) \cap \prod(A\diamondsuit_1 S\diamondsuit_1 A)]$ is the generalized 2-left bi-quasi ideal generated by a set A in S.*

Theorem 4.4. *For any $a \in S$, the 2-right bi-quasi ideal generated by "a", denoted by $< a >_{2rbq}$ is given by $\{a^n | n \in \mathbb{Z}^+\} \cup [(a\diamondsuit_1 S) \cap (a\diamondsuit_1 S\diamondsuit_1 a)]$.*

Theorem 4.5. *For any $a \in S$, the 2-left bi-quasi ideal generated by "a", denoted by $< a >_{2lbq}$ is given by $\{a^n | n \in \mathbb{Z}^+\} \cup [(S\diamondsuit_1 a) \cap (a\diamondsuit_1 S\diamondsuit_1 a)]$.*

Corollary 4.3. *For the subset A of S, $\prod A \cup [\prod(A\diamondsuit_1 S) \cap \prod(S\diamondsuit_1 A)]$ is the 2-right bi-quasi ideal generated by a set A in S.*

Theorem 4.6. *For a b-semiring S, the following statements are equivalent.*
(1) S is 2-regular.
(2) $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$, for the weak-2-right ideals R and generalized 2-right bi-quasi ideals Q_1.
(3) $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$, for the weak-2-right ideals R and 2-right bi-quasi ideals Q_1.
(4) $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$, for the weak-2-right ideals R and generalized 2-left bi-quasi ideals Q_1.
(5) $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$, for the weak-2-right ideals R and 2-left bi-quasi ideals Q_1.
(6) $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$, for the weak-2-right ideals R and generalized 2-quasi ideals Q_1.
(7) $R \cap Q \subseteq R\diamondsuit_1 Q$, for the weak-2-right ideals R and 2-quasi ideals Q.
(8) $Q_1 \cap L \subseteq Q_1\diamondsuit_1 L$, for the generalized 2-right bi-quasi ideals Q_1 and weak-2-left ideals L.
(9) $Q_1 \cap L \subseteq Q_1\diamondsuit_1 L$, for the generalized 2-left bi-quasi ideals Q_1 and weak-2-left ideals L.
(10) $Q_1 \cap L \subseteq Q_1\diamondsuit_1 L$, for the generalized 2-quasi ideals Q_1 and weak-2-left ideals L.

(11) $Q \cap L \subseteq Q\diamondsuit_1 L$, for the 2-quasi ideals Q and weak-2-left ideals L.
(12) $R \cap L = R\diamondsuit_1 L$, for the weak-2-right ideals R and weak-2-left ideals L.

Proof: First, we prove that (1) ⇒ (2) ⇒ (3) ⇒ (7) ⇒ (12), (1) ⇒ (8) ⇒ (10) ⇒ (11) ⇒ (12) ⇒ (1), (1) ⇒ (4) ⇒ (5) ⇒ (7) and (2) ⇒ (6) ⇒ (12), (1) ⇒ (9) ⇒ (10) ⇒ (12) ⇒ (1).
(1) ⇒ (2) For $a \in R \cap Q_1$, then there exists $s \in S$ such that $a = (a\diamondsuit_1 s)\diamondsuit_1 a$. Thus, $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$.
(2) ⇒ (3) By Remark 4.1, the result holds.
(3) ⇒ (7) By Theorem 4.1, then (7) follows.
(7) ⇒ (12) By Remark 4.2, $R \cap L \subseteq R\diamondsuit_1 L$. Now, $R\diamondsuit_1 L \subseteq R\diamondsuit_1 S \subseteq R$ and $R\diamondsuit_1 L \subseteq S\diamondsuit_1 L \subseteq L$. Therefore, $R\diamondsuit_1 L \subseteq R \cap L$. Hence, $R\diamondsuit_1 L = R \cap L$. Then (12) follows.
(1) ⇒ (8) For $a \in Q_1 \cap L$, then there exists $s \in S$ such that $a = a\diamondsuit_1(s\diamondsuit_1 a)$. Thus, $Q_1 \cap L \subseteq Q_1\diamondsuit_1 L$.
(8) ⇒ (10) The Proof follows from Theorem 4.1.
(10) ⇒ (11) The proof follows from Theorem Lemma 4.4 [18], and the result holds.
(11) ⇒ (12) The proof follows from Theorem 4.10 [18].
(1) ⇒ (4) For $a \in R \cap Q_1$, then there exists $s \in S$ such that $a = (a\diamondsuit_1 s)\diamondsuit_1 a$. Thus, $R \cap Q_1 \subseteq R\diamondsuit_1 Q_1$.
(4) ⇒ (5) By Theorem 4.1, then (5) follows.
(5) ⇒ (7) The proof follows from Theorem 4.1.
(2) ⇒ (6) By Theorem 4.1, the proof follows.
(6) ⇒ (12) The proof follows from Theorem 4.10 [18], and the result holds.
(1) ⇒ (9) For $a \in Q_1 \cap L$, then there exists $s \in S$ such that $a = a\diamondsuit_1(s\diamondsuit_1 a)$. Thus, $Q_1 \cap L \subseteq Q_1\diamondsuit_1 L$.
(9) ⇒ (10) By Theorem 4.1, the proof follows.
(10) ⇒ (12) By Remark 4.2, then (12) holds.
(12) ⇒ (1) The proof follows from Theorem 3.16 [18]. □

Theorem 4.7. *For a b-semiring S, the following statements are equivalent.*
(1) S is 2-regular.
(2) $B_1 \cap I \cap B_2 \subseteq B_1\diamondsuit_1 I\diamondsuit_1 B_2$, for the generalized bi-quasi ideals B_1 and B_2 and weak-2 ideals I.
(3) $B \cap I \cap Q \subseteq B\diamondsuit_1 I\diamondsuit_1 Q$, for the generalized bi-quasi ideals B, weak-2 ideals I and 2-quasi ideals Q.
(4) $Q \cap I \cap B \subseteq Q\diamondsuit_1 I\diamondsuit_1 B$, for the 2-quasi ideals Q, weak-2 ideals I and generalized bi-quasi ideals B.
(5) $B \cap I \cap B \subseteq B\diamondsuit_1 I\diamondsuit_1 B$, for the bi-quasi ideals B and weak-2 ideals I.
(6) $B_1 \cap I \cap L \subseteq B_1\diamondsuit_1 I\diamondsuit_1 L$, for the generalized 2-bi-quasi ideals B_1, weak-2 ideals I and weak-2-left ideals L.
(7) $B \cap I \cap L \subseteq B\diamondsuit_1 I\diamondsuit_1 L$, for the bi-quasi ideals B, weak-2 ideals I and weak-2-left ideals L.
(8) $R \cap I \cap B_2 \subseteq R\diamondsuit_1 I\diamondsuit_1 B_2$, for the weak-2-right ideals R, weak-2 ideals I and generalized bi-quasi ideals B_2.
(9) $R \cap I \cap B \subseteq R\diamondsuit_1 I\diamondsuit_1 B$, for the weak-2-right ideals R, weak-2 ideals I and bi-quasi ideals B.
(10) $R \cap I \cap L \subseteq R\diamondsuit_1 I\diamondsuit_1 L$, for the weak-2-right ideals R, weak-2 ideals I and weak-2-left ideals L.
(11) $R \cap L = R\diamondsuit_1 L$, for the weak-2-right ideals R and weak-2-left ideals L.
(12) $B_1 \cap I \subseteq B_1\diamondsuit_1 I\diamondsuit_1 B_1$, for the generalized bi-quasi ideals B_1 and weak-2 ideals I.
(13) $B \cap I \subseteq B\diamondsuit_1 I\diamondsuit_1 B$, for the bi-quasi ideals B and weak-2 ideals I.

(14) $B_1 = B_1 \diamondsuit_1 S \diamondsuit_1 B_1$*, for the generalized bi-quasi ideals* B_1*.*
(15) $Q = Q \diamondsuit_1 S \diamondsuit_1 Q$*, for the 2-quasi ideals* Q*.*

Proof: First, we prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (1), (2) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (2) $\Rightarrow$ (5) $\Rightarrow$ (9) $\Rightarrow$ (15) $\Rightarrow$ (1), (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (10), (2) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (15) $\Rightarrow$ (1) and (12) $\Rightarrow$ (14) $\Rightarrow$ (15) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) For $a \in Q_1 \cap I \cap Q_2$, then there exists $s \in S$ such that $a = a \diamondsuit_1 s \diamondsuit_1 a$. Thus, $a = a \diamondsuit_1 (s \diamondsuit_1 a \diamondsuit_1 s) \diamondsuit_1 a \in Q_1 \diamondsuit_1 I \diamondsuit_1 Q_2$. Thus (2) holds.
(2) $\Rightarrow$ (3) Straightforward.
(3) $\Rightarrow$ (7) The proof follows from Theorem 4.10 [18].
(7) $\Rightarrow$ (10) By Remark 4.2, then (10) holds.
(10) $\Rightarrow$ (11) Taking $I = S$ in (5), $R \cap L \subseteq R \diamondsuit_1 L$. For weak-2-right ideal R and weak-2-left ideal L, $R \diamondsuit_1 L \subseteq R \diamondsuit_1 S \subseteq R$ and $R \diamondsuit_1 L \subseteq S \diamondsuit_1 L \subseteq L$. Therefore $R \diamondsuit_1 L \subseteq R \cap L$. Thus, $R \cap L = R \diamondsuit_1 L$.
(11) $\Rightarrow$ (1) The proof follows from Theorem 3.16 [18].
(2) $\Rightarrow$ (4) Straightforward.
(4) $\Rightarrow$ (8) The proof follows from Theorem 4.10 [18], and we get the result.
(8) $\Rightarrow$ (9) Straightforward.
(9) $\Rightarrow$ (10) By Remark 4.2, then (10) holds.
(2) $\Rightarrow$ (5) Straightforward.
(5) $\Rightarrow$ (9) By Remark 4.2, the proof follows.
(9) $\Rightarrow$ (15) Straightforward.
(1) $\Rightarrow$ (6) For $a \in B_1 \cap I \cap L$, then there exists $s \in S$ such that $a = a \diamondsuit_1 s \diamondsuit_1 a$. Thus, $a = a \diamondsuit_1 (s \diamondsuit_1 a \diamondsuit_1 s) \diamondsuit_1 a \in B_1 \diamondsuit_1 I \diamondsuit_1 L$. Thus, (6) holds.
(6) $\Rightarrow$ (7) Straightforward.
(2) $\Rightarrow$ (12) Taking $B_2 = B_1$ in (2), we get the result.
(12) $\Rightarrow$ (13) Straightforward.
(13) $\Rightarrow$ (15) Straightforward.
(12) $\Rightarrow$ (14) Taking $I = S$, $B_1 \subseteq B_1 \diamondsuit_1 S \diamondsuit_1 B_1 \subseteq [(B_1 \diamondsuit_1 S) \cap (S \diamondsuit_1 B_1)] \subseteq B_1$ implies $B_1 = B_1 \diamondsuit_1 S \diamondsuit_1 B_1$.
(14) $\Rightarrow$ (15) Straightforward.
(15) $\Rightarrow$ (1) For any $a \in S$ by (15), $a \in < a >_{1q} \diamondsuit_1 S \diamondsuit_1 < a >_{1q}$ and by Theorem 4.5 and Lemma 4.1. Thus, $a \in a \diamondsuit_1 S \diamondsuit_1 a$. Hence, S is 2-regular. □

5. **Conclusion.** Several characterizations of the 1-bi-quasi ideal (2-bi-quasi ideal) of b-semirings are described in this article. Our discussion has focused on some of their fundamental characteristics and has also examined some of them using the various bi-quasi ideals and their generators. We presented the 1-bi-quasi ideal (2-bi-quasi ideal) of b-semirings, which was constructed from b-semirings based on element and subset. At the end of our discussion, we explored the relationship between quasi-ideals and bi-quasi ideals. In the future, we plan to explore a few more types of tri-ideals and prime bi-ideals. Our study will examine their research on hyper b-semirings using bi-ideals and bi-quasi ideals.

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