

A DIVERSITY BASED ON POPULATION SEARCH DOMAIN FOR TWO-SWARM COMPETITIVE OPTIMIZATION ALGORITHM

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ABSTRACT. *Diversity analysis plays a critical role in quantifying the efficiency of particle swarm optimization algorithm, which can be used to improve the global search ability. Traditional definitions ignore evaluating the zone being searched repeatedly, which means that these definitions may not describe the actual search range of particles. Here, different from the classical diversity definitions, we decompose diversity into two parts including the dispersion of axis and the average amplitude. The dispersion of the axis represents the spatial distribution of the particles, while the average amplitude represents the average size of the particle search range. To filter out the zone unable to be searched effectively, the diversity includes the confidence setting. Then, an evolutionary mechanism is established based on optimum and diversity. Two-swarm competitive optimization algorithm (TCPSO) is proposed based on newly defined diversity. The experiments show that the new diversity definition can help the algorithm get the optimal solution effectively. In experiments for solving systems of nonlinear equations, TCPSO exhibited an order of magnitude advantage on multiple test equation systems. We have discovered a previously undiscovered solution to a set of nonlinear equations based on TCPSO.*

Keywords: Diversity analysis, Particle swarm optimization algorithm, Two-swarm competitive optimization algorithm, Systems of nonlinear equations

1. Introduction. Particle swarm optimization (PSO) was firstly proposed by Kennedy and Eberhart in [1]. It has been widely used because of simple iterative formula, easy programming features and good optimization performance [2]. PSO and its variants have been used in several fields, such as engineering structure design [3], resource scheduling [4], medical image [5], automatic control [6], economics [7] and machine learning optimization [8,9].

As a population-based algorithm, the representation of diversity is critical. Diversity is divided into position diversity and velocity diversity. Position diversity is calculated according to the position information of the swarm, and velocity diversity is calculated according to the velocity information of the swarm. In addition, the diversity is divided into element-wise and dimension-wise according to the form of diversity in [10]. Shi and Eberhart gave a position diversity, which averages the distance in all dimensions, smoothing out the diversity in a single dimension, and losing some effective information

[11]. To obtain a more reasonable representation of particle diversity, Krink et al. proposed a similar definition of diversity in [12], which considered the diversity of particles in each dimension individually. This definition works well and has been applied in [12-15]. However, this definition only considers the relationship between particles and the central particle rather than the relationship between each pair of particles.

In order to better describe the position diversity, Shi and Eberhart put forward a new definition in [10] based on the definition proposed in [16,17], which considered the distance between each particle. Olorunda and Engelbrecht proposed another definition in [19], which also calculates the distance between each pair of particles. Moreover, Pant et al. normalized the diversity using maximum distance between particles in [16]. In addition to the study of position diversity, Shi and Eberhart proposed a velocity diversity in [10], which studies the diversity of particle velocities in different dimensions.

Previous researches have proposed various methods for the swarm to jump out of the local optimal solution [19-22]. Maintaining population diversity has become a critical method to make swarm jump out of local optimal solutions [24]. Tanweer et al. proposed to divide the variants of PSO into four categories [25]. This classification can also be used in the methods of increasing diversity. The four categories are highlighted below.

Algorithms based on parameter settings: Inertia weight and acceleration coefficients are the most important parameters in PSO. Inertia weight settings can be divided into deterministic setting and random setting. Deterministic setting includes linearly changing [26] and nonlinearly changing [27]. The randomness setting includes Gaussian distribution setting [28], fuzzy adaptive setting [29] and self-regulating setting [25]. The researches on acceleration coefficient setting include expanded PSO (XPSO) [30], and adaptive weighting PSO (AWPSO) [31]. All the above parameter settings make the population sacrifice the convergence speed to delay the decay of the diversity of the population, which shows the importance of diversity for the algorithm to search for the global optimal solution.

Algorithms based on topology: Kennedy proposed that the topology of particles has a huge impact on the performance of the algorithm [32]. In PSO, all particles are attracted to previous best position and global best position, which causes the population to get stuck in local optima solution due to the rapid decrease in diversity [24]. Zhang et al. proposed a comprehensive learning particle swarm optimizer with local optima topology [33]. Zou et al. proposed a close neighbor mobility optimization algorithm (CNMM) based on the ring neighborhood topology [34]. Li et al. promoted information exchange between different particles by scaling the neighborhood [35]. Lazzús et al. added a random guide particle to change the topology, which provides new information to the population [36]. These topologies allow information about general particles in the population to be recorded, which increases the diversity of the population [37].

Algorithms based on hybridized version: Although PSO has a fast convergence, it may fall into local minima easily due to the rapid decline in diversity [38]. To ameliorate this deficiency, the researchers introduced other strategies into PSO [39]. Chu et al. proposed the shuffled complex evolution with principal components analysis – University of California at Irvine (SP-UCI) based on the shuffled complex evolution – University of Arizona method (SCE-UA) [40]. SP-UCI maintains the population's diversity in update processing based on the capability of searching over the full search space. Mahmoodabadi et al. proposed high exploration particle swarm optimization (HEPSO) based on multi-crossover mechanism and bee colony mechanism [41]. HEPSO maintains the population diversity based on multi-crossover mechanism, which is also applied in genetic algorithm (GA). Khan and Ling proposed hybrid gravitational search particle swarm optimization algorithm (HGSPSO) based on gravitational search algorithm (GSA) [42], which

uses GSA to delay the decline in population diversity. Ding and Gu increased the search range of individual particles to enhance the population diversity [43]. Compared to PSO, whether the above methods detect diversity or not, they all increase the diversity of the population in the calculation process.

Algorithms based on learning strategies: Traditional PSO does not pay attention to the change of population diversity, which results in the fact that the algorithm cannot well balance exploration and exploitation [19]. To balance exploration and exploitation, Yang et al. maintained population diversity through particle mutation in [44]. Tao et al. improved the diversity of the group through the mutual cooperation of the two particle swarms [46]. Borowska proposed a learning competitive swarm optimization (LCSO), which updates the optimal particle through competition among multiple swarms to maintain the diversity of the group and ensure the convergence speed [47]. In addition, Li et al. proposed a pyramid PSO (PPSO), which maintains diversity through cooperation and competition among multiple subgroups [48]. Li et al. classified particles based on velocity, allowing different particles to perform different search tasks [49]. The above methods all utilize various iterative strategies to balance exploration and exploitation.

Although many types of diversity definitions have emerged and been applied in [9,11,12,14], none of these definitions describe the effective search range of the particles and the population. Some algorithms increase population diversity by adding randomness [23,45], essentially increasing the effective search range of particles. Some algorithms utilize multiple subpopulations to increase population diversity [46,50], which essentially is improving the effective search range of the population. In addition, Liu et al. proposed multi-technique diversity-based particle swarm optimization, which implements different search strategy (strong exploration ability or strong exploitation ability) based on the population diversity [51]. However, nobody gives a quantitative relationship between exploration and exploitation.

This paper proposes a new diversity definition based on the description about the effective search range of the particles and the population. The new definition consists of two parts: the dispersion of the particles in the population when iterating and the average search range of the particles. The former represents the effective search range of the population, and the other one represents the average effective search range of the particles. TCPSO introduces confidence coefficient, which we can adjust to calculate diversity more aggressively or moderately. Different from the traditional method of setting thresholds, TCPSO introduces a competition mechanism, which takes position and velocity mutation operations in different iteration periods by comparing the optimum and diversity values of the two swarms. We search the global search ability of TCPSO based on numerical experiments.

In this paper, the diversities of two particle swarms will be continuously adjusted via the competition of two particle swarms based on newly defined diversity. In Section 2, we introduce the new diversity definition TCPSO. In Section 3, we search the effectiveness of the new diversity definition and TCPSO. In Section 4, some conclusions are provided.

Nomenclature

Latin		Latin		Latin	
$\mathbf{X}_n^{(t)}$	Position	$\mathbf{V}_n^{(t)}$	Velocity	w	Inertia weight
c_1	Individual acceleration	c_2	Social acceleration	$P_{n,pbest}$	Individual best position
P_{gbest}	Best position	r_1, r_2	Uniformly distributed variables	α	$c_1(P_{n,pbest}^t - X_{nd}^{(t)})$
β	$c_2(P_{gbest}^t - X_{nd}^{(t)})$	γ	$r_1\alpha + r_2\beta$	k	Confidence
$\mathbf{A}^d$	List of axes	$\overline{\mathbf{A}^d}$	Sorted list of axes	$\mathbf{A}_L^d$	Equidistance list of axes
$\widetilde{\mathbf{A}^d}$	$\mathbf{A}_L^d - \overline{\mathbf{A}^d}$	Da	Dispersion of axes	$\mathbf{Am}$	Average amplitude
Div	Diversity	m_D	Mutation rate of the velocity	m_A	Mutation rate of the position
θ	Variable of judging stage				

2. Method. PSO is an intelligent algorithm for simulating bird foraging, which includes a swarm of particles. The position of each particle in the population corresponds to a potential solution, and the position of each particle is updated by the update formula. The n -th particle at the t -th iteration has a position $\mathbf{X}_n^{(t)} = (x_{n1}^{(t)}, x_{n2}^{(t)}, \dots, x_{nD}^{(t)})$ and a velocity $\mathbf{V}_n^{(t)} = (v_{n1}^{(t)}, v_{n2}^{(t)}, \dots, v_{nD}^{(t)})$ in D dimensional space, and the velocity is confined within $[V_{\min}, V_{\max}]^D$. The update formulas of PSO are

$$V_{nd}^{(t+1)} = w \cdot V_{nd}^{(t)} + c_1 \cdot r_1 \cdot (P_{n,pbest}^t - X_{nd}^{(t)}) + c_2 \cdot r_2 \cdot (P_{gbest}^t - X_{nd}^{(t)}), \quad (1)$$

$$X_{nd}^{(t+1)} = X_{nd}^{(t)} + V_{nd}^{(t+1)}, \quad (2)$$

where w denotes the inertia weight; c_1 denotes the individual acceleration; c_2 denotes the social acceleration; $P_{n,pbest}^t$ denotes the best position of the n -th particle after t iterations; P_{gbest}^t denotes the best position of all the particles after t iterations; r_1 and r_2 are uniformly distributed in $[0, 1]$.

2.1. New definition of diversity. In order to investigate the search range and dispersion of particles in the iterative process, we need to investigate the update formula of particles, which has not been found in other diversity definitions. Equations (1) and (2) can be combined and written as

$$X_{nd}^{(t+1)} = X_{nd}^{(t)} + wV_{nd}^{(t)} + r_1\alpha + r_2\beta, \quad (3)$$

$$\alpha = c_1 (P_{n,pbest}^t - X_{nd}^{(t)}), \quad (4)$$

$$\beta = c_2 (P_{gbest}^t - X_{nd}^{(t)}). \quad (5)$$

Here the probability distribution of $X_{nd}^{(t+1)}$ depends on $r_1\alpha$ and $r_2\beta$. Obviously $r_1\alpha$ and $r_2\beta$ are random variables with uniform distribution. Let $\gamma = r_1\alpha + r_2\beta$. As shown in Equation (3), the value of $X_{nd}^{(t)} + wV_{nd}^{(t)}$ is determined, while the value of $r_1\alpha + r_2\beta$ is random. We will study $r_1\alpha + r_2\beta$ to judge the possible search region of particles and the probability distribution of particle arrival in each region. The distributions of γ under different conditions are shown in Equations (A1)-(A8) (refer to Appendix A).

From Equations (A1)-(A8) (refer to Appendix A), we find that the probability density of γ is symmetrical and the probability distribution of γ depends on α and β . For example, if $\alpha > 0$, $\beta > 0$ and $\alpha > \beta$, the probability density of γ is

$$f(\gamma = t) = \begin{cases} \frac{t}{\alpha\beta} & t \in (0, \beta) \\ \frac{1}{\alpha} & t \in (\beta, \alpha) \\ \frac{-t + \alpha + \beta}{\alpha\beta} & t \in (\alpha, \alpha + \beta) \end{cases}. \quad (6)$$

When $t \in (\beta, \alpha)$, the probability density is a constant $1/\alpha$, which means γ is uniformly distributed in $[\beta, \alpha]$. The axis of symmetry is $\frac{\alpha+\beta}{2}$. The proof process is shown in Appendix B.

Like the normal distribution, this paper introduces the concept of confidence. We use k ($0 \leq k \leq 1$) to present confidence degree. In this paper, the confidence degree k represents the probability that the random variable γ falls in a certain interval during the particle updating process. By introducing k , we give priority to regions with higher probability density when calculating the range of γ . If $\alpha > 0$, $\beta > 0$ and $\alpha > \beta$, the interval of γ based on k is as shown below.

If $k \leq 1 - \frac{\beta}{\alpha}$,

$$\int_{\frac{\alpha+\beta}{2}-\varepsilon}^{\frac{\alpha+\beta}{2}+\varepsilon} \frac{1}{\alpha} dt = k, \quad (7)$$

where $0 \leq \varepsilon \leq \frac{\alpha-\beta}{2}$. We get

$$\varepsilon = \frac{\alpha k}{2}. \quad (8)$$

Then $\gamma \in \left(\frac{\alpha+\beta}{2} - \frac{\alpha k}{2}, \frac{\alpha+\beta}{2} + \frac{\alpha k}{2}\right)$. If $k > 1 - \frac{\beta}{\alpha}$,

$$\int_{\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}}^{\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}} \frac{1}{\alpha} dt + 2 \int_{\frac{\alpha+\beta}{2}-\varepsilon}^{\beta} \frac{t}{\alpha\beta} dt = k. \quad (9)$$

We get

$$\varepsilon = \frac{\alpha + \beta}{2} - \sqrt{\alpha\beta(1-k)}, \quad (10)$$

where $\frac{\alpha-\beta}{2} < \varepsilon \leq \frac{\alpha+\beta}{2}$. Then $\gamma \in \left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}\right), \frac{\alpha+\beta}{2} + \left(\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}\right)\right)$. Like above analysis, the interval of γ is listed in Table 1 under all conditions.

We can calculate the search range of the n -th particle in the d -th dimension with a certain confidence k in the t -th iteration according to Table 1. Especially, optimal particle has an equal probability of reaching any point within its searchable range when α or β is equal to 0.

We will derive a new calculation method of particle diversity based on the above analysis. Equation (3) will be rewritten as

$$X_{nd}^{(t+1)} = X_{nd}^{(t)} + wV_{nd}^{(t)} + \frac{\alpha + \beta}{2} + \left(\gamma - \frac{\alpha + \beta}{2}\right). \quad (11)$$

Here, $X_{nd}^{(t)} + wV_{nd}^{(t)} + \frac{\alpha+\beta}{2}$ represents the axis of the search range of the n -th particle in the d -th dimension in the t -th iteration and $\gamma - \frac{\alpha+\beta}{2}$ represents the amplitude. We will define diversity based on the axis and amplitude. The proposed diversity consists of the axis dispersion and the average amplitude. The calculation process of the dispersion of axis in the d -th dimension will be listed below. Firstly, we get the list of axes

$$\mathbf{A}^d = [a_1^d, a_2^d, \dots, a_n^d, \dots, a_N^d], \quad (12)$$

where a_n^d denotes the axis of the n -th particle in the d -th dimension. N denotes the number of particles. We sort $\mathbf{A}^d$

$$\overline{\mathbf{A}^d} = [\min(\mathbf{A}^d), \dots, \max(\mathbf{A}^d)]. \quad (13)$$

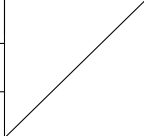
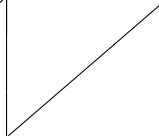
A set of particles that were equidistance in the d dimension would have such a set $\mathbf{A}_L^d$ of axes:

$$\mathbf{A}_L^d = \left[\min(\mathbf{A}^d), \dots, \min(\mathbf{A}^d) + \frac{\max(\mathbf{A}^d) - \min(\mathbf{A}^d)}{N-1} \times (i-1), \dots, \max(\mathbf{A}^d) \right], \quad (14)$$

where i denotes the index of the axis. We get

$$\widetilde{\mathbf{A}^d} = \mathbf{A}_L^d - \overline{\mathbf{A}^d}. \quad (15)$$

TABLE 1. The interval of γ under different α , β and k

α, β		k	The interval of γ
$\alpha > 0, \beta > 0$	$\alpha \geq \beta$	$k \leq 1 - \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - \frac{\alpha k}{2}, \frac{\alpha+\beta}{2} + \frac{\alpha k}{2})$
		$k > 1 - \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}), \frac{\alpha+\beta}{2} + (\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}))$
	$\alpha < \beta$	$k \leq 1 - \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - \frac{\beta k}{2}, \frac{\alpha+\beta}{2} + \frac{\beta k}{2})$
		$k > 1 - \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - (\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}), \frac{\alpha+\beta}{2} + (\frac{\alpha+\beta}{2} - \sqrt{\alpha\beta(1-k)}))$
$\alpha > 0, \beta < 0$	$\alpha \geq \beta $	$k \leq 1 + \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - \frac{\alpha k}{2}, \frac{\alpha+\beta}{2} + \frac{\alpha k}{2})$
		$k > 1 + \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (\frac{\alpha-\beta}{2} - \sqrt{\alpha\beta(k-1)}), \frac{\alpha+\beta}{2} + (\frac{\alpha-\beta}{2} - \sqrt{\alpha\beta(k-1)}))$
	$\alpha < \beta $	$k \leq 1 + \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} + \frac{\beta k}{2}, \frac{\alpha+\beta}{2} - \frac{\beta k}{2})$
		$k > 1 + \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - (\frac{\alpha-\beta}{2} - \sqrt{\alpha\beta(k-1)}), \frac{\alpha+\beta}{2} + (\frac{\alpha-\beta}{2} - \sqrt{\alpha\beta(k-1)}))$
$\alpha < 0, \beta > 0$	$ \alpha \geq \beta$	$k \leq 1 + \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (-\frac{\alpha k}{2}), \frac{\alpha+\beta}{2} + (-\frac{\alpha k}{2}))$
		$k > 1 + \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (\frac{\beta-\alpha}{2} - \sqrt{\alpha\beta(k-1)}), \frac{\alpha+\beta}{2} + (\frac{\beta-\alpha}{2} - \sqrt{\alpha\beta(k-1)}))$
	$ \alpha < \beta$	$k \leq 1 + \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - (-\frac{\beta k}{2}), \frac{\alpha+\beta}{2} - (-\frac{\beta k}{2}))$
		$k > 1 + \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - (\frac{\beta-\alpha}{2} - \sqrt{\alpha\beta(k-1)}), \frac{\alpha+\beta}{2} + (\frac{\beta-\alpha}{2} - \sqrt{\alpha\beta(k-1)}))$
$\alpha < 0, \beta < 0$	$ \alpha \geq \beta $	$k \leq 1 - \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (-\frac{\alpha k}{2}), \frac{\alpha+\beta}{2} - (-\frac{\alpha k}{2}))$
		$k > 1 - \frac{\beta}{\alpha}$	$(\frac{\alpha+\beta}{2} - (\frac{-\alpha-\beta}{2} - \sqrt{\alpha\beta(1-k)}), \frac{\alpha+\beta}{2} + (\frac{-\alpha-\beta}{2} - \sqrt{\alpha\beta(1-k)}))$
	$ \alpha < \beta $	$k \leq 1 - \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - \frac{\beta k}{2}, \frac{\alpha+\beta}{2} + \frac{\beta k}{2})$
		$k > 1 - \frac{\alpha}{\beta}$	$(\frac{\alpha+\beta}{2} - (\frac{-\alpha-\beta}{2} - \sqrt{\alpha\beta(1-k)}), \frac{\alpha+\beta}{2} + (\frac{-\alpha-\beta}{2} - \sqrt{\alpha\beta(1-k)}))$
$\alpha = 0, \beta = 0$			0
$\alpha = 0, \beta \neq 0$			$(\min(0, \beta), \max(0, \beta))$
$\alpha \neq 0, \beta = 0$			$(\min(0, \alpha), \max(0, \alpha))$

Then the dispersion of axes in the d -th dimension Da^d is

$$Da^d = \frac{\|\widetilde{\mathbf{A}^d}\|_2}{N \times (\max(\mathbf{A}^d) - \min(\mathbf{A}^d))}. \quad (16)$$

The amplitude of the n -th particle in the d -th dimension amp_n^d is if $\alpha > 0$, $\beta > 0$, $\alpha > \beta$ and $k > 1 - \frac{\beta}{\alpha}$,

$$amp_n^d = \alpha + \beta - 2\sqrt{\alpha\beta(1-k)}. \quad (17)$$

The average amplitude $\mathbf{Am}$ in the d -th dimension is

$$Am^d = \frac{1}{N} \sum_{n=1}^N amp_n^d. \quad (18)$$

The diversity of a swarm Div is

$$Div^d = \left[\frac{\|\mathbf{A}_L^d - \overline{\mathbf{A}^d}\|_2}{N \times (\max(\mathbf{A}^d) - \min(\mathbf{A}^d))}, \frac{1}{N} \sum_{n=1}^N (2\gamma - \alpha - \beta) \right], \quad (19)$$

where $Da^d = \frac{\|\mathbf{A}_L^d - \overline{\mathbf{A}^d}\|_2}{N \times (\max(\mathbf{A}^d) - \min(\mathbf{A}^d))}$, $Am^d = \frac{1}{N} \sum_{n=1}^N (2\gamma - \alpha - \beta)$. Different from the commonly used diversity definitions, the diversity Div^d consists of two parts including the axis dispersion Da^d and the average amplitude Am^d in the d -th dimension. Da^d represents the spatial distribution of the particle's potential search range in the d -th dimension. The smaller Da^d , the larger the search area is searched by the particles in the d -th dimension. Am^d represents the average size of the particle search range in the d -th dimension. The larger Am^d means the larger the average range of particle search in the d -th dimension. The classical position diversity is

$$\bar{x} = \frac{1}{D \times N} \sum_{d=1}^D \sum_{n=1}^N x_{dn}, \quad (20)$$

$$Div^p = \frac{1}{D \times N} \sum_{d=1}^D \sum_{n=1}^N (x_{dn} - \bar{x})^2. \quad (21)$$

From Equation (21), we find that the search range in the subsequent iterations of particles does not focus on the current position but will be symmetrically distributed around the axis $X_{nd}^{(t)} + wV_{nd}^{(t)} + \frac{\alpha+\beta}{2}$. We set the confidence degree k to investigate the effective search range of particles in a certain confidence interval. For example, it means that the probability of particles appearing in a certain region in the next iteration reaches 90% when $k = 0.9$. The setting of confidence degree filters the areas where particles are rarely likely to appear, which can ensure the stability of diversity calculation, referring to Figure 1.

2.2. Two-swarm competitive optimization algorithm. Unlike PSO, two-swarm competitive optimization algorithm (TCPSO) has two independent iterative particle swarms. In PSO, we try to ensure that particles have strong exploration ability in the exploration stage and strong exploitation ability in the exploitation stage. In TCPSO, we do not pay attention to the optimal values of swarms in the exploration stage, and we only compare the diversities of the two swarms and mutate the swarm with low diversity. In the exploitation stage, we give priority to comparing the optimal values of the two swarms and mutate the particle swarm with high optimal value and low diversity. In

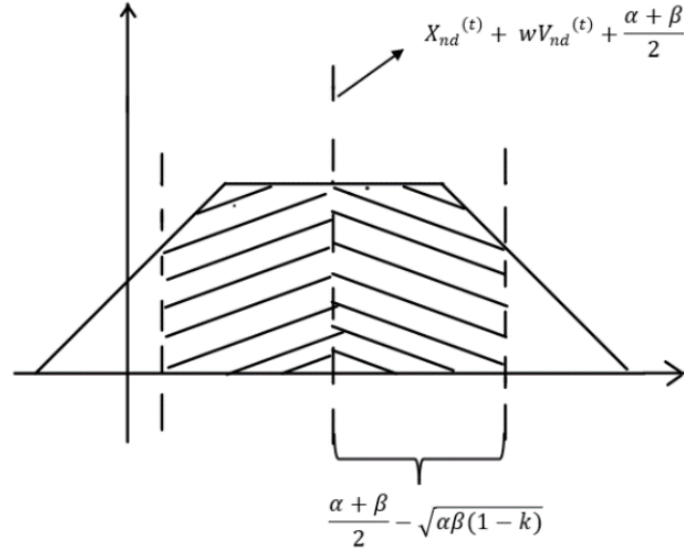


FIGURE 1. Shaded areas represent values that the particle has a high probability of obtaining, and non-shaded areas represent values that are less likely to be obtained by particles. $\alpha > 0$, $\beta > 0$, $\alpha \geq \beta$, $k > 1 - \frac{\beta}{\alpha}$, the axis of symmetry is $X_{nd}^{(t)} + wV_{nd}^{(t)} + \frac{\alpha + \beta}{2}$, the effective search range along the axis of symmetry is $\frac{\alpha + \beta}{2} - \sqrt{\alpha\beta(1-k)}$, and the proportion of the shaded area to the total area is k . k represents the cumulative probability distribution of particle values.

order to ensure the universality of the method, we adopt uniformly distributed particle initialization [52]. Algorithm flow chart is shown in Figure 2.

The pseudocode incorporating the hyperparameters used in the manuscript is provided below.

- 1) Initialize S_1 and S_2 . The population size $N = 50$.
- 2) If $t \leq 0.8 \times 1000$, compare the particle distribution interval of S_1 and S_2 with the mean search amplitude. Apply velocity mutation to the swarm with smaller particle distribution interval and position mutation to the swarm with smaller mean search amplitude, where the mutation rates are 0.05. Update S_1 and S_2 .
- 3) If $t > 0.8 \times 1000$, first compare the fitness values of S_1 and S_2 . Select the swarm with higher fitness value and perform velocity mutation or position mutation based on the comparison results of particle distribution intervals and mean search amplitudes between the two swarms, where the mutation rates are 0.05.
- 4) Repeat step 3 until the iteration termination criterion is satisfied, and then terminate the program.

Here, θ is a parameter that controls the competition rules of particle swarms. The large θ means that the swarms enter the exploitation stage late, and the competition of swarms focus on diversity. Currently, the algorithm has strong exploration ability, which is conducive to the swarm to jump out of the local optimal solution. The small θ means that swarms enter the exploitation stage early, and the competition of swarms focuses on the optimal value. Currently, the algorithm has strong exploitation ability, which is conducive to the swarms to obtain higher precision solutions. In the exploration stage, we will directly compare the diversities of the two swarms, and then mutate the swarm with lower diversity to maintain the ability of efficient global exploration. In the exploitation stage, we will first compare the optimal solutions of the two swarms, then compare the

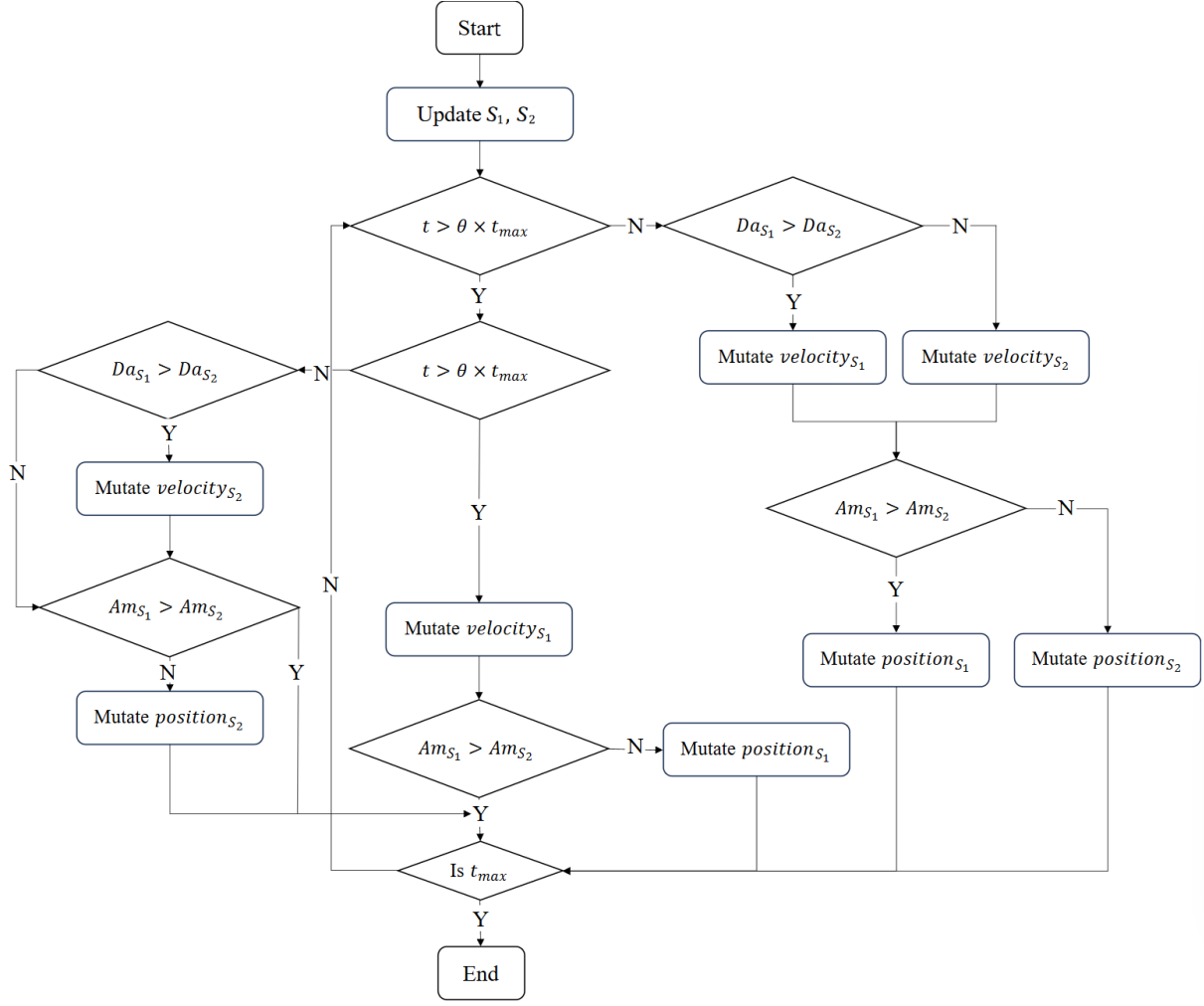


FIGURE 2. m_D denotes the mutation rate of the velocity, and m_A denotes the mutation rate of the position. In the exploration phase, the algorithm preferentially compares the diversity of the two particle swarms, performs velocity mutation operation on the swarm with smaller axial dispersion, and performs position mutation on the swarm with smaller average amplitude. When the algorithm enters the exploitation stage, the algorithm first compares the optimum of the two swarms, and then compares the diversity of the two swarms. In this stage, algorithm performs velocity mutation for the swarm with a larger optimum and a smaller axial dispersion and performs position mutation for the swarm with a larger optimum and a smaller average amplitude. The parameter θ is used to judge the time when the algorithm is in the exploitation stage. Here, $\theta \in (0, 1)$.

diversities of the two swarms, and finally mutate the swarm with high optimal value and poor diversity to improve the accuracy of the algorithm.

Div^d consists of two parts: the dispersion of axis Da^d and the average amplitude Am^d . In Equation (21), Da^d depends on the position and velocity of the particles, while Am^d depends on the position of the particles. We determine the mutation strategies based on Div^d . When it is necessary to improve Da^d , we will mutate the velocity of some particles in the population, and the mutation rate is m_D . When we need to improve Am^d , we will mutate the position of some particles in the population, and the mutation rate is m_A .

3. Numerical Experiments and Discussion. In order to test TCPSO, we compared the optimization results of PSO and TCPSO. We use two definitions of diversity in TCPSO, one is the classical definition of diversity, referring to Equation (21), and the other one is the new diversity definition, referring to Equation (19). Six test functions in these experiments are shown in Table 2. There are many local optimal solutions in these six test functions, and their two-dimensional images are shown in Figure 3. The iterative convergence curves of PSO, TCPSO with classical definition and TCPSO with new definition under the same parameter setting are shown in Figure 4. The change frequency of their optimal value is shown in Figure 5. In this paper, the parameters θ and k in TCPSO are also tested. The experimental results are shown in Figures 6-8. The mutation rate was fixed at 0.05 for all algorithms. The maximum velocity is constrained to 1/20 of the range between the upper and lower boundaries.

TABLE 2. Six functions are tested, where x^* is the optimal solution.

Function	$f(x)$	Theoretical value	x^*	Range
f_1	$\sum_{i=1}^n (x_i^2 - 10 \cos(2\pi x_i) + 10)$	0	$[0, 0, \dots, 0]$	$[-10, 10]$
f_2	$\frac{1}{4000} \sum_{i=1}^n x_i^2 - \prod_{i=1}^n \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$	0	$[0, 0, \dots, 0]$	$[-600, 600]$
f_3	$-\frac{1 + \cos\left(\frac{12\sqrt{x_1^2 + x_2^2}}{0.5(x_1^2 + x_2^2) + 2}\right)}{0.5(x_1^2 + x_2^2) + 2}$	-1	$[0, 0]$	$[-5.12, 5.12]$
f_4	$-(x_2 + 47) \sin\left(\sqrt{\left x_2 + \frac{x_1}{2} + 47\right }\right) - x_1 \sin\left(\sqrt{\left x_1 - (x_2 + 47)\right }\right)$	-959.6407	$[512, 404.2319]$	$[-512, 512]$
f_5	$10d + \sum_{i=1}^d [x_i^2 - 10 \cos(2\pi x_i)]$	0	$[0, 0, \dots, 0]$	$[-5.12, 5.12]$
f_6	$418.9829d - \sum_{i=1}^d x_i \sin\left(\sqrt{ x_i }\right)$	0	$[420.9687, \dots, 420.9687]$	$[-500, 500]$

As shown in Figure 3, the test functions have many local minimums, which means that the algorithm needs the ability to jump out of the local optimal solution. f_3 and f_4 are fixed two-dimensional test functions. f_1 , f_2 , f_5 and f_6 can be extended to any dimensions. The minimum values of f_1 , f_2 , f_3 and f_5 are located in the center of the images, and the minimum values of f_4 and f_6 are located at the edge of the images.

In the comparative experiment of algorithm optimization ability, we test each algorithm for 10 times and draw the iterative curves. It can be seen in Figure 4(b) that the three optimization methods have good optimization ability in f_2 . As presented in Figures 4(a) and 4(e), PSO converges to the local optimal solution for many times on f_1 and f_5 and cannot jump out, while TCPSO achieves the theoretical minimum value of the experimental function, which means that TCPSO has a stronger ability to jump out of the local optimal solution. As illustrated in Figures 4(c), 4(d) and 4(f), both PSO and TCPSO using the classical diversity definition cannot jump out of the local optimal solution in a large number of cases, while TCPSO using the new diversity definition jump out of the local optimal solution with a high probability, which means that the new diversity definition can better help the population jump out of the local optimal solution on f_3 , f_4 and f_6 compared with the classical diversity definition.

In order to further search the new diversity definition, we make a comparison of the frequency diagram of the optimum of PSO, TCPSO using the classical diversity definition

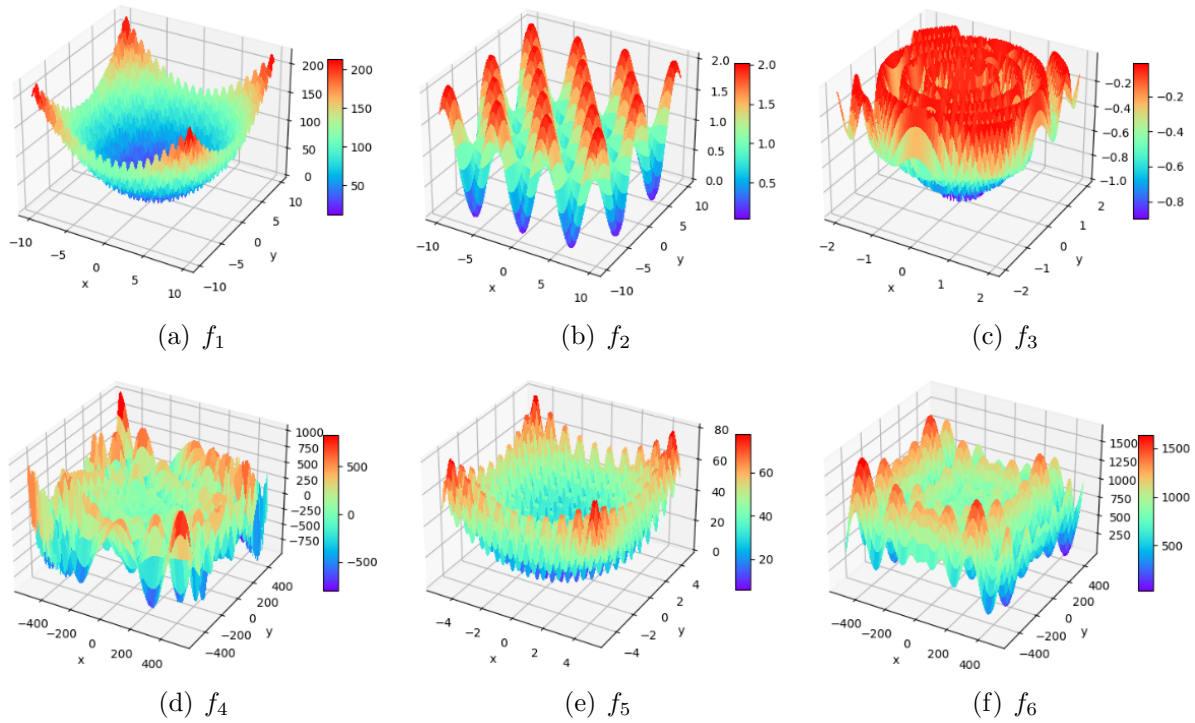


FIGURE 3. f_3 and f_4 are fixed two-dimensional test functions, and f_1 , f_2 , f_5 and f_6 can be extended to any dimensions. The minimum values of f_1 , f_2 , f_3 and f_5 are located in the center of the images, and the minimum values of f_4 and f_6 are located at the edge of the images.

and TCPSO using the new diversity definition on six test functions. As illustrated in Figures 5(a), 5(b), 5(d), 5(e) and 5(f), when the iteration reaches the maximum value, the change frequency of the optimum of TCPSO is much higher than that of PSO, which means that TCPSO has higher global search potential. As shown in Figure 5(c), the change frequencies of the optimum in three methods are similar and low due to the availability of global optimal solution. As presented in Figures 5(b), 5(d), 5(e) and 5(f), the TCPSO using the new diversity definition has a significantly higher change frequency of the optimum compared with the TCPSO using the classical diversity definition, which means that the new diversity definition can better describe the diversity of particle swarm and help particles better jump out of the local optimal solution. In Figure 5(a), the TCPSO using the new diversity definition has the same optimum change frequency as the TCPSO using the classical diversity definition, but the TCPSO using the new diversity definition has higher stability, which further proves that the new diversity definition has higher potential in describing the diversity of particle swarm.

In the exploration stage, we need to maintain high particle diversity as much as possible. In the exploitation stage, we need to speed up the convergence speed of particle swarm to make the particles converge to the optimal solution faster. Therefore, the judgment in the iterative stage is very important for the final optimization result of particle swarm. θ is the parameter used to judge the iteration stage in TCPSO. The larger θ means that the algorithm will give priority to the diversity of particle swarm in more iterations to prevent premature aggregation of particles. The small θ means that the algorithm will enter the fast convergence stage early. The algorithm will not consider the diversity of particles when $\theta = 0$. The algorithm will always give priority to the diversity of particles when $\theta = 1$.

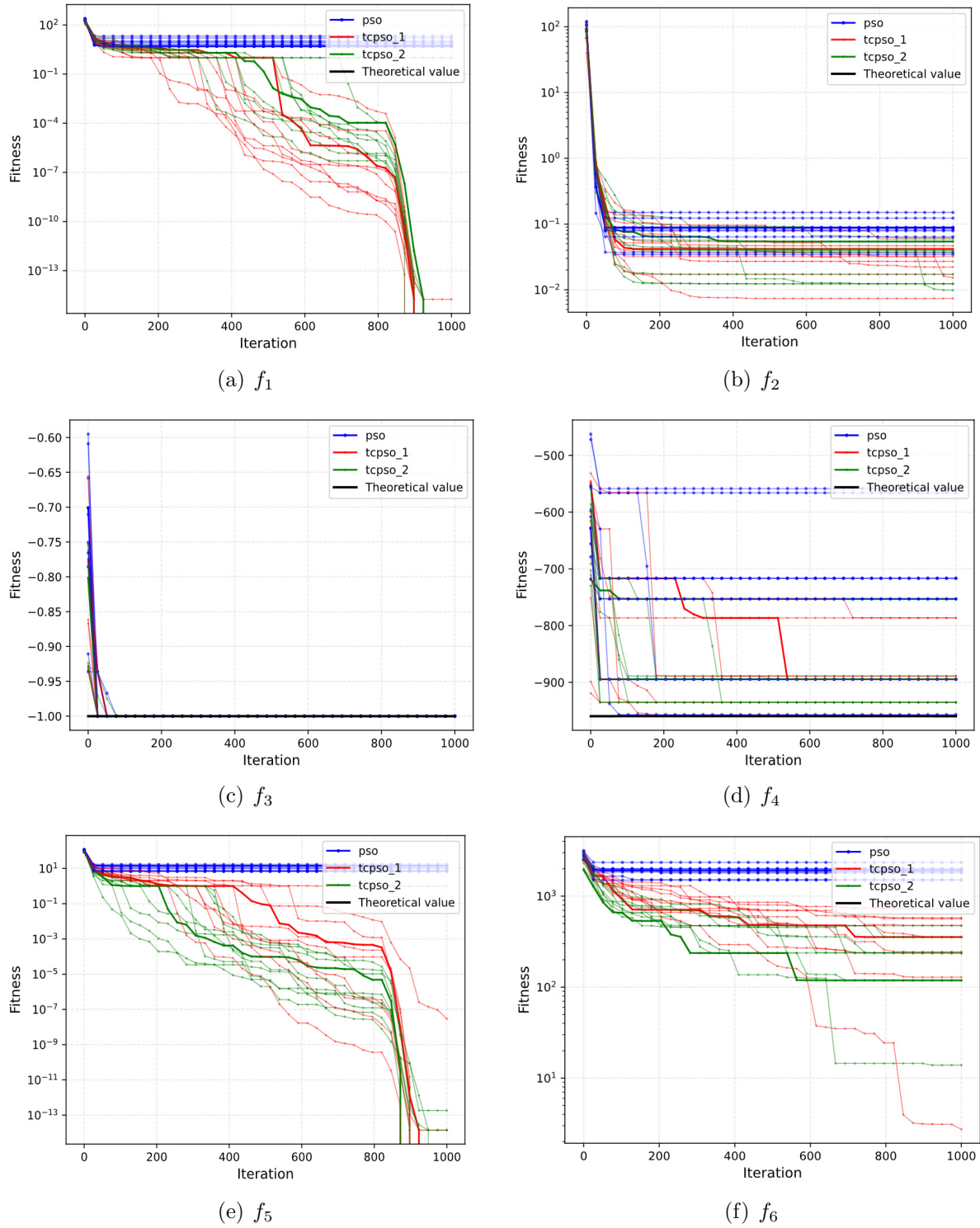


FIGURE 4. (color online) The curves of the best optimum values of PSO, TCPSO using the classical diversity (tcpso_1) and TCPSO using the new diversity definition (tcpso_2) on the six test functions with iterations are compared with the theoretical optimal values as shown above. Each function is tested 10 times. Each swarm has 50 particles in three algorithms. The stopping criterion of three algorithms is to reach 1000 times. The dimensions of $f_1, f_2, f_3, f_4, f_5, f_6$ are $10d, 10d, 2d, 2d, 10d, 10d$. Here $k = 0.5$, $\theta = 0.8$.

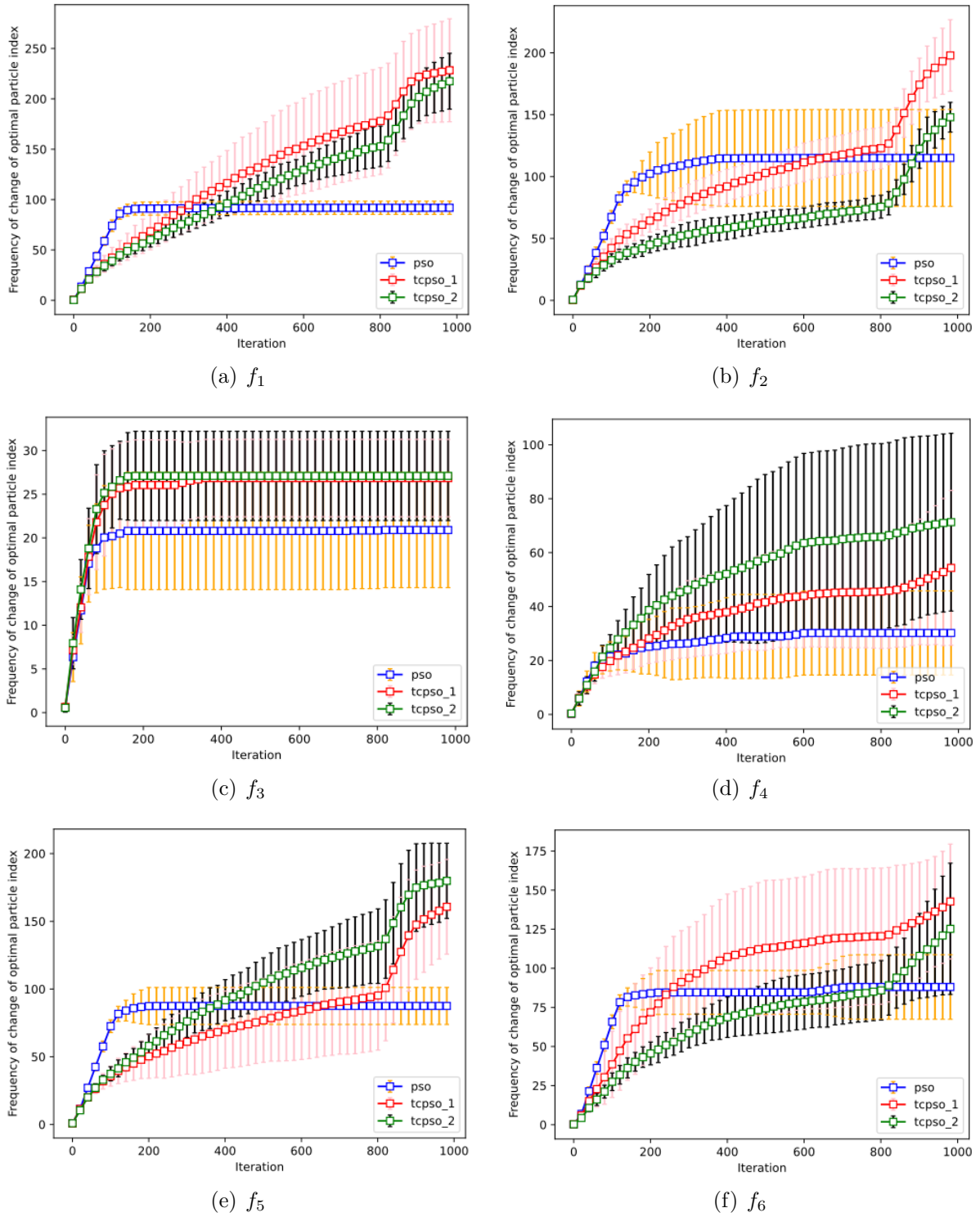


FIGURE 5. (color online) The curves of the optimum index change frequency of PSO, TCPSO using the classical diversity (tcpso_1) and TCPSO using the new diversity definition (tcpso_2) on the six test functions as a function of iteration. Each swarm has 50 particles in three algorithms. The stopping criterion of three algorithms is to reach 1000 times. The dimensions of f_1 , f_2 , f_3 , f_4 , f_5 , f_6 are $10d$, $10d$, $2d$, $2d$, $10d$, $10d$. Here $k = 0.5$, $\theta = 0.8$.

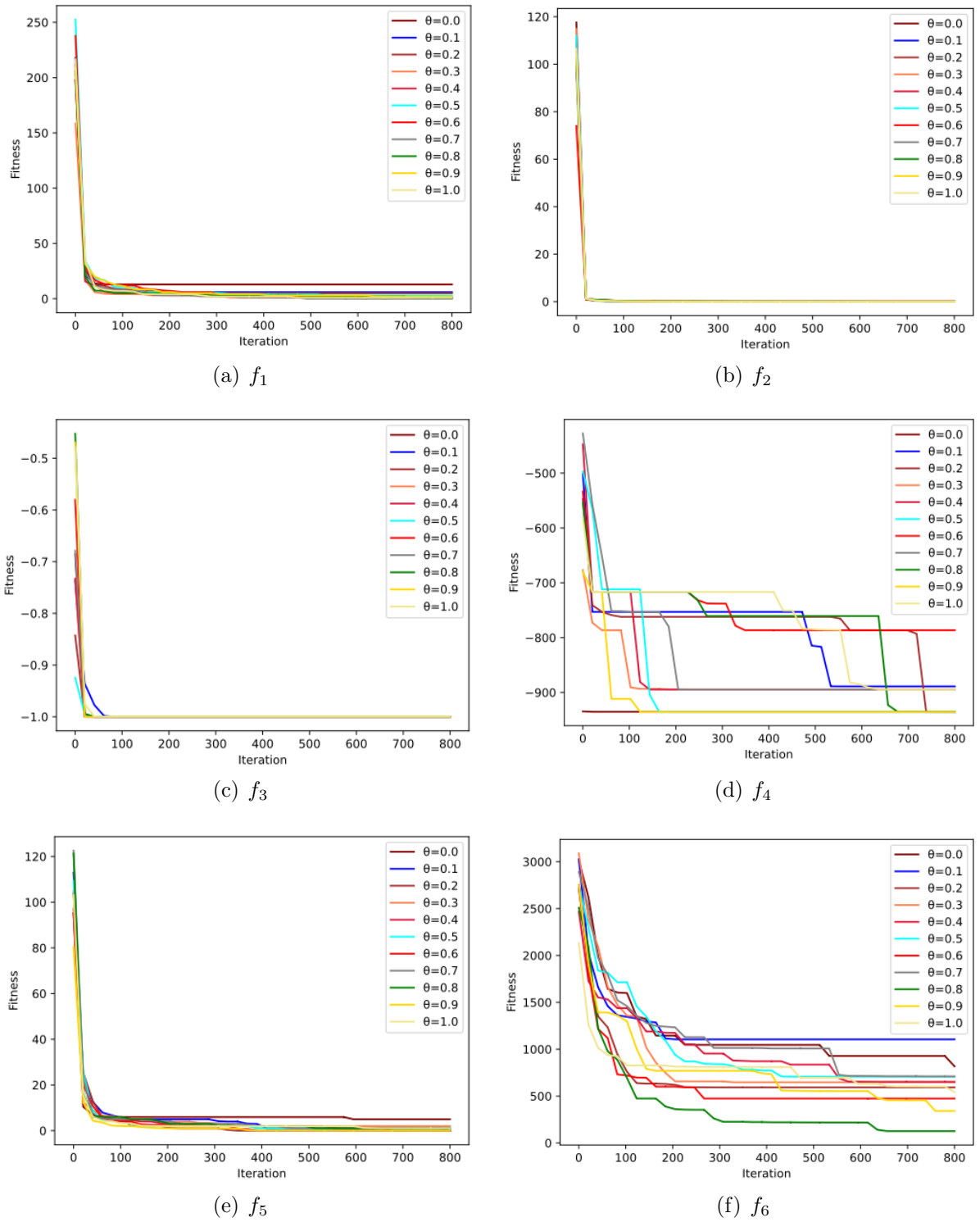


FIGURE 6. (color online) When $\theta = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, the iterative convergence curve of the algorithm. Each swarm has 50 particles. The dimensions of $f_1, f_2, f_3, f_4, f_5, f_6$ are $10d, 10d, 2d, 2d, 10d, 10d$. The stopping criterion is to reach specific times. Here $k = 0.5$.

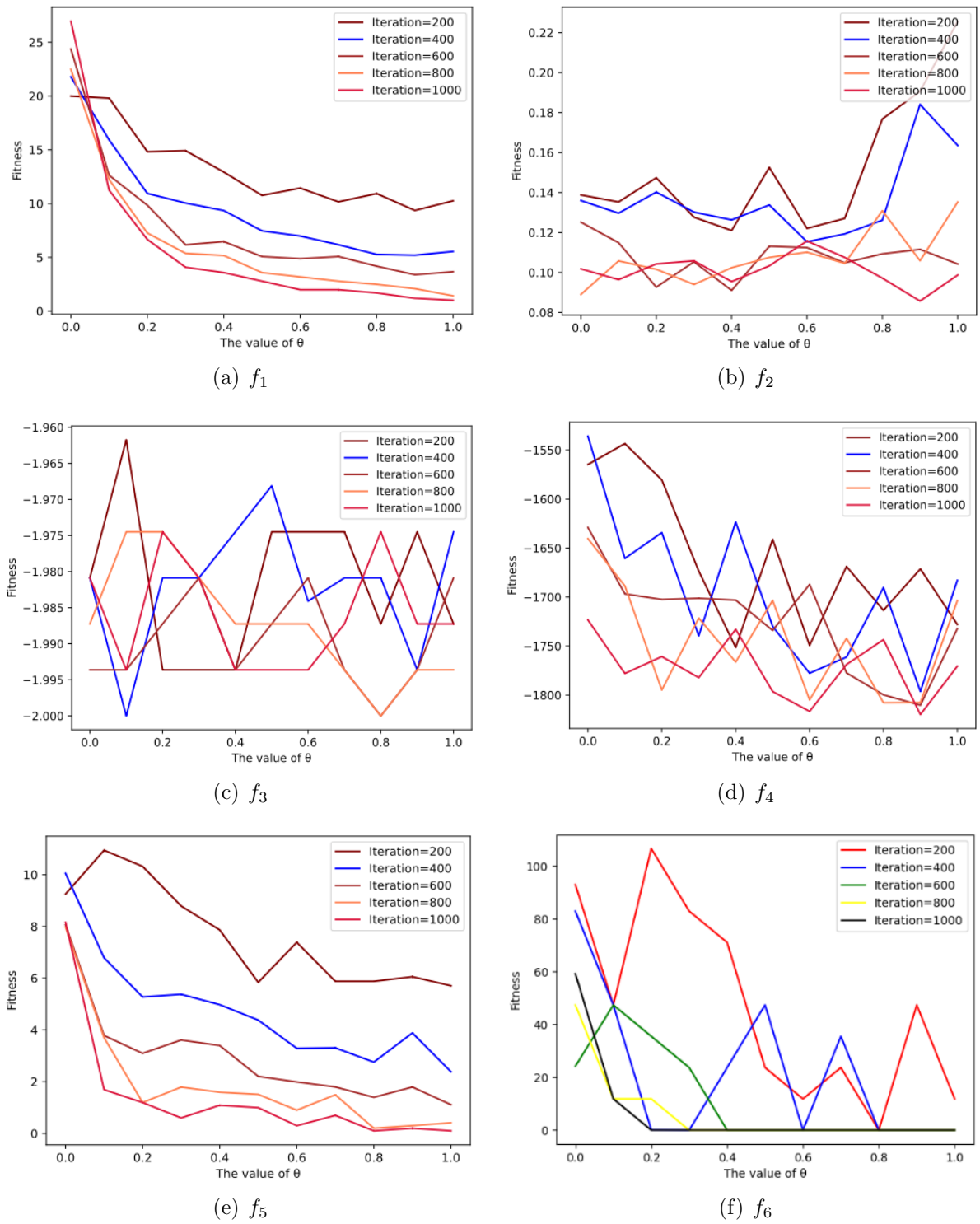


FIGURE 7. (color online) The best optimum under different iteration times $iteration = \{200, 400, 600, 800, 1000\}$ and $\theta = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$. Each group of parameters is tested 20 times, and optimum is the average value of 20 tests. Each swarm has 50 particles. The dimensions of $f_1, f_2, f_3, f_4, f_5, f_6$ are $10d, 10d, 2d, 2d, 10d, 10d$. The stopping criterion is to reach specific times. Here $k = 0.5$.

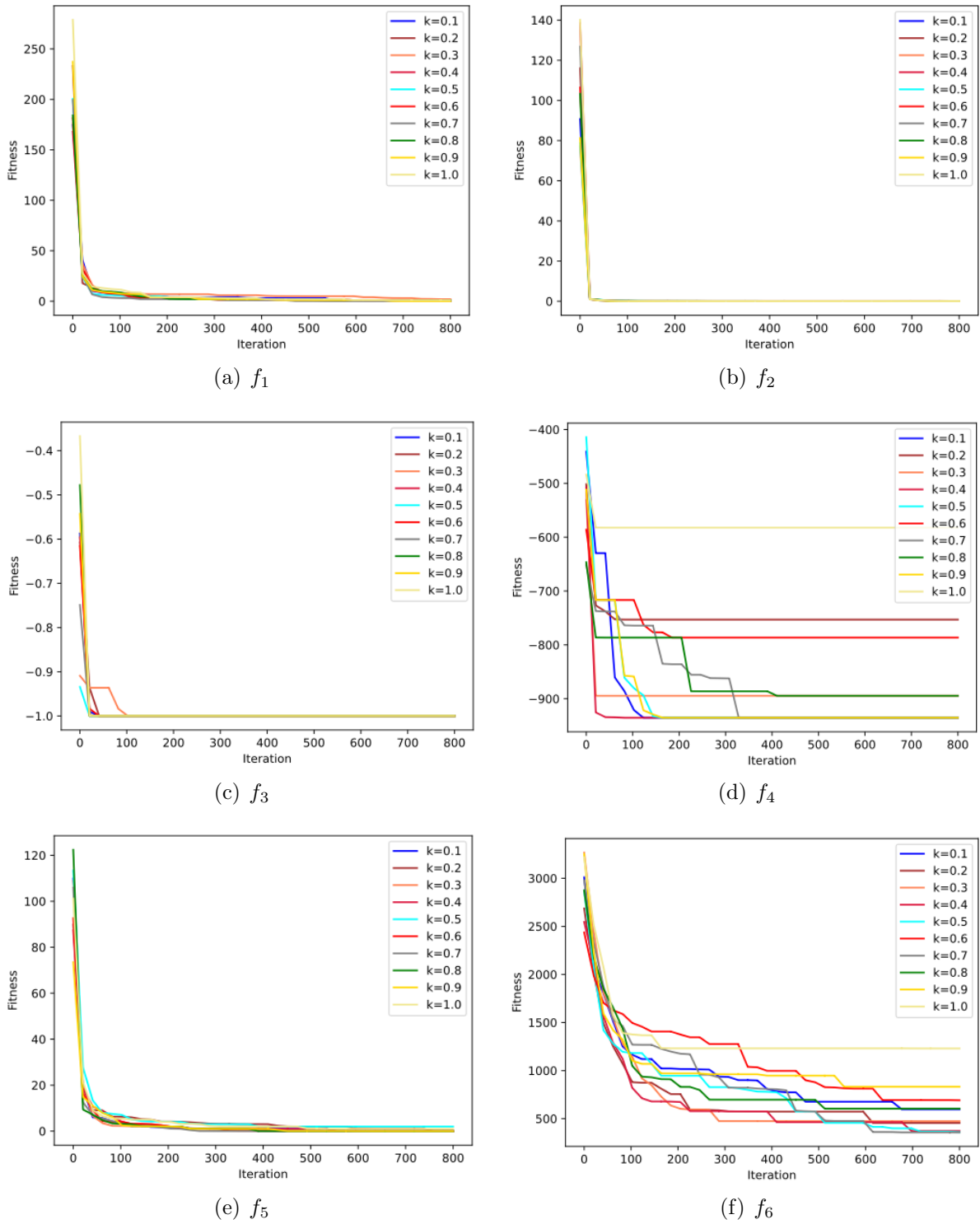


FIGURE 8. (color online) When $k = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, the iterative convergence curve of the algorithm. Each swarm has 50 particles. The dimensions of $f_1, f_2, f_3, f_4, f_5, f_6$ are $10d, 10d, 2d, 2d, 10d, 10d$. The stopping criterion is to reach specific times. Here $\theta = 0.8$.

As shown in Figure 6, we draw the iterative curve of the algorithm under different θ . The particle swarm converges to the local optimal solution prematurely and fails to jump out all the time when $\theta = 0$. Contrary to our intuition, particles fail to converge to the global optimal solution when $\theta = 1$. In fact, TCPSO has the greatest potential to converge to the global optimum when $\theta = 0.8$. The above phenomenon shows that in the late stage of convergence, if we give priority to population diversity, the population will search in a large range and cannot converge to the global optimum.

In order to search the influence of *iteration* and θ on the optimal value of the TCPSO, we test the influence of different $\theta = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ on the optimum when *iteration* = $\{200, 400, 600, 800, 1000\}$. The experimental results show that no matter what value *iteration* is taken, the algorithm falls into the local optimal solution when $\theta < 0.2$, which is consistent with the experimental results in Figure 7. In addition, we find that when optimizing the function with multiple local minimum values, a large *iteration* may lead to a waste of computing resources. Different θ has little effect on the final optimum of the algorithm when *iteration* ≥ 800 and $\theta > 0.2$. If *iteration* < 800 , different θ has a great impact on the optimum of the algorithm, which can be ignored when $\theta = 0.8$. It is also consistent with the experimental results in Figure 7. It is worth mentioning that the optimum of the algorithm decreases with θ increasing when *iteration* < 800 , which means that a large θ can improve the algorithm with few computing resources.

The influence of parameter k on TCPSO was investigated using six test functions. As illustrated in Figure 8, the results demonstrate that intermediate values of k yield robust performance, with the parameter exhibiting no drastic impact on the algorithm's efficacy. Consequently, a value of $k = 0.5$ is recommended.

The optimization results of CSO, CLPSO, DE, and TCPSO were benchmarked across six test functions, as summarized in Table 3. The experimental outcomes demonstrate that TCPSO achieves a significant advantage on four of the six functions. This validates that the new diversity definition enhances the population's potential for locating the global optimum.

TABLE 3. The comparison of the best fitness values obtained by the four methods (CSO, CLPSO, DE and TCPSO) on the six test functions

Function	CSO	CLPSO	DE	TCPSO
f_1	$3.71\text{e-}6 \pm 6.22\text{e-}6$	$1.41\text{e-}11 \pm 7.23\text{e-}12$	$3.92\text{e-}5 \pm 2.73\text{e-}5$	0 ± 0
f_2	$1.35\text{e-}1 \pm 5.68\text{e-}1$	$3.44\text{e-}2 \pm 2.53\text{e-}2$	$6.39\text{e-}2 \pm 3.42\text{e-}2$	$1.44\text{e-}2 \pm 4.97\text{e-}2$
f_3	0 ± 0	0 ± 0	0 ± 0	0 ± 0
f_4	$5.61\text{e-}6 \pm 3.44\text{e-}5$	$4.44\text{e-}3 \pm 7.92\text{e-}3$	$6.73\text{e-}4 \pm 2.99\text{e-}4$	$3.73\text{e-}5 \pm 6.98\text{e-}5$
f_5	$1.43\text{e-}6 \pm 3.76\text{e-}6$	$1.42\text{e-}14 \pm 4.55\text{e-}13$	$1.42\text{e-}14 \pm 1.33\text{e-}13$	0 ± 0
f_6	$3.79\text{e+}3 \pm 2.14\text{e+}3$	$1.44\text{e+}2 \pm 1.32\text{e+}2$	$4.75\text{e+}3 \pm 5.28\text{e+}3$	$7.26\text{e+}1 \pm 6.32\text{e+}1$

4. Conclusions. In this paper, a new diversity definition for a particle swarm $Div^d = \left[\frac{\|A_L^d - \bar{A}^d\|_2}{N \times (\max(A^d) - \min(A^d))}, \frac{1}{N} \sum_{n=1}^N (2\gamma - \alpha - \beta) \right]$ is proposed based on the dynamic information of particles. In the experiment, we find that the new definition of diversity can better describe the diversity of the particle swarm, so that the particle swarm has a greater chance of jumping out of the local optimal solution in the iterative process. A new iterative strategy TCPSO is established based on the new definition of diversity, and numerical experiments show that this strategy can improve the global search ability after using the new definition of diversity. TCPSO can balance the exploration and exploitation of the

algorithm and we give a recommended value $\theta = 0.8$. In the application of solving systems of nonlinear equations, TCPSO has shown remarkable solving ability and discovered a previously undiscovered set of solutions.

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Appendix A. If $\alpha > 0$, $\beta > 0$ and $\alpha > \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{t^2}{2\alpha\beta} & t \in (0, \beta) \\ \frac{2t - \beta}{2\alpha} & t \in (\beta, \alpha) \\ \frac{2\alpha\beta - (\alpha + \beta - t)^2}{2\alpha\beta} & t \in (\alpha, \alpha + \beta) \end{cases} . \quad (\text{A1})$$

If $\alpha > 0$, $\beta > 0$ and $\alpha \leq \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{t^2}{2\alpha\beta} & t \in (0, \alpha) \\ \frac{2t - \alpha}{2\beta} & t \in (\alpha, \beta) \\ \frac{2\alpha\beta - (\alpha + \beta - t)^2}{2\alpha\beta} & t \in (\beta, \alpha + \beta) \end{cases} . \quad (\text{A2})$$

If $\alpha > 0$, $\beta < 0$ and $\alpha > |\beta|$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \beta)^2}{-2\alpha\beta} & t \in (\beta, 0) \\ \frac{2t - \beta}{2\alpha} & t \in (0, \alpha + \beta) \\ \frac{2\alpha\beta + (\alpha - t)^2}{2\alpha\beta} & t \in (\alpha + \beta, \alpha) \end{cases} . \quad (\text{A3})$$

If $\alpha > 0$, $\beta < 0$ and $\alpha \leq |\beta|$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \beta)^2}{-2\alpha\beta} & t \in (\beta, \alpha + \beta) \\ \frac{2t - 2\beta - \alpha}{-2\beta} & t \in (\alpha + \beta, 0) \\ \frac{2\alpha\beta + (\alpha - t)^2}{2\alpha\beta} & t \in (0, \alpha) \end{cases} . \quad (\text{A4})$$

If $\alpha < 0$, $\beta > 0$ and $|\alpha| > \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \alpha)^2}{-2\alpha\beta} & t \in (\alpha, \alpha + \beta) \\ \frac{2t - 2\alpha - \beta}{-2\alpha} & t \in (\alpha + \beta, 0) \\ \frac{2\alpha\beta + (\beta - t)^2}{2\alpha\beta} & t \in (0, \beta) \end{cases} . \quad (\text{A5})$$

If $\alpha < 0$, $\beta > 0$ and $|\alpha| \leq \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \alpha)^2}{-2\alpha\beta} & t \in (\alpha, \alpha + \beta) \\ \frac{2t - \alpha}{-2\beta} & t \in (\alpha + \beta, 0) \\ \frac{2\alpha\beta + (\beta - t)^2}{2\alpha\beta} & t \in (0, \beta) \end{cases} . \quad (\text{A6})$$

If $\alpha < 0$, $\beta < 0$ and $|\alpha| > \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \alpha - \beta)^2}{-2\alpha\beta} & t \in (\alpha + \beta, \alpha) \\ \frac{2t - 2\alpha - \beta}{-2\alpha} & t \in (\alpha, \beta) \\ \frac{2\alpha\beta - t^2}{2\alpha\beta} & t \in (\beta, 0) \end{cases} . \quad (\text{A7})$$

If $\alpha < 0$, $\beta < 0$ and $|\alpha| \leq \beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \alpha - \beta)^2}{-2\alpha\beta} & t \in (\alpha + \beta, \beta) \\ \frac{2t - 2\alpha - \beta}{-2\alpha} & t \in (\beta, \alpha) \\ \frac{2\alpha\beta - t^2}{2\alpha\beta} & t \in (\alpha, 0) \end{cases} . \quad (\text{A8})$$

If $\alpha > 0$, $\beta < 0$ and $\alpha = -\beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t - \beta)^2}{-2\beta^2} & t \in (\beta, 0) \\ \frac{2\beta^2 + (\beta + t)^2}{2\beta^2} & t \in (0, \alpha) \end{cases} . \quad (\text{A9})$$

If $\alpha < 0$, $\beta > 0$ and $\alpha = -\beta$, the probability distribution of γ is

$$P(\gamma \leq t) = \begin{cases} \frac{(t + \beta)^2}{2\beta^2} & t \in (\alpha, 0) \\ \frac{-2\beta^2 + (\beta - t)^2}{-2\beta^2} & t \in (0, \beta) \end{cases} . \quad (\text{A10})$$

The derivation of Equation (A1) is shown in Figure A1.

The specific form of the PDF depends on the relationship between α and β .

Subcase 1.1: Symmetric Coefficients $\alpha = \beta$

Support: $[0, 2\alpha]$; Distribution: Symmetric Triangular Distribution; The PDF is symmetric about its midpoint $\gamma = \alpha$, where it attains its maximum value of $\frac{1}{\alpha}$.

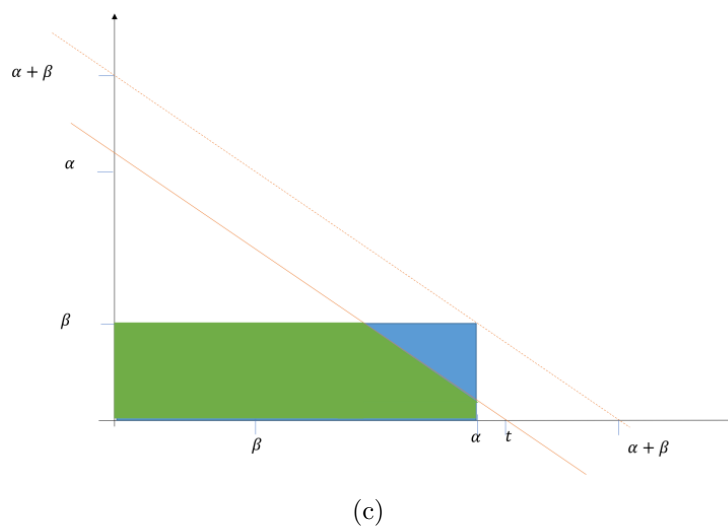
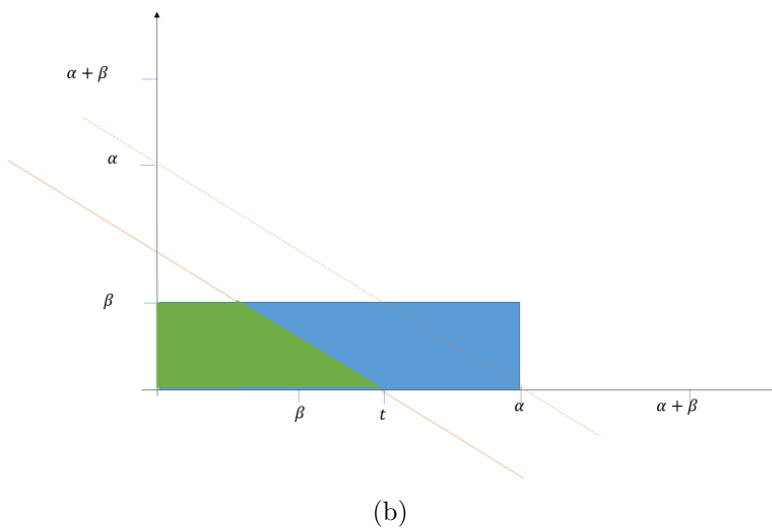
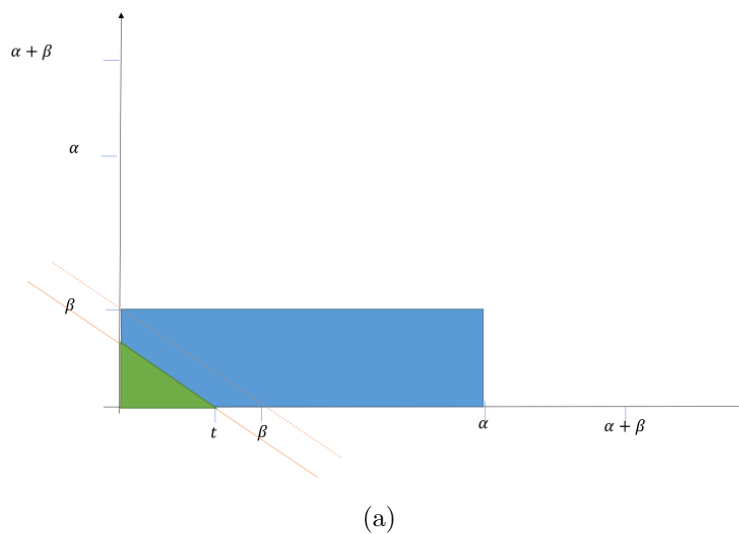


FIGURE A1. (color online) As shown in Figure A1(a), here $t \in (0, \beta)$, $P(\gamma \leq t)$ is equal to the size of the green area. As shown in Figure A1(b), here $t \in (\beta, \alpha)$, $P(\gamma \leq t)$ is equal to the size of the green area. As shown in Figure A1(c), here $t \in (\alpha, \alpha + \beta)$, $P(\gamma \leq t)$ is equal to the size of the green area.

PDF:

$$f(\gamma = t) = \begin{cases} \frac{t}{\alpha^2} & t \in (0, \alpha) \\ \frac{2\alpha - t}{\alpha^2} & t \in (\alpha, 2\alpha) \\ 0 & \text{otherwise} \end{cases} \quad (\text{A11})$$

Subcase 1.2: Asymmetric Coefficients $\alpha < \beta$

Support: $[0, \alpha + \beta]$; Distribution: Trapezoidal Distribution; The PDF is characterized by three distinct segments: a linear increase, a constant plateau, and a linear decrease.

PDF:

$$f(\gamma = t) = \begin{cases} \frac{t}{\alpha\beta} & t \in (0, \alpha) \\ \frac{1}{\beta} & t \in (\alpha, \beta) \\ \frac{\alpha + \beta - t}{\alpha\beta} & t \in (\beta, \alpha + \beta) \\ 0 & \text{otherwise} \end{cases} \quad (\text{A12})$$

Subcase 1.3: Asymmetric Coefficients $\alpha > \beta$

Support: $[0, \alpha + \beta]$; Distribution: Trapezoidal Distribution; This is the symmetric counterpart to Subcase 1.2.

PDF:

$$f(\gamma = t) = \begin{cases} \frac{t}{\alpha\beta} & t \in (0, \beta) \\ \frac{1}{\alpha} & t \in (\beta, \alpha) \\ \frac{\alpha + \beta - t}{\alpha\beta} & t \in (\alpha, \alpha + \beta) \\ 0 & \text{otherwise} \end{cases} \quad (\text{A13})$$

Case 2: Negative Coefficients $\alpha < 0, \beta < 0$. Let $\alpha' = -\alpha, \beta' = -\beta$.

Support: $[\beta + \alpha, 0]$; Distribution: Symmetric Triangular (if $\alpha = \beta$) or Trapezoidal Distribution. The PDF is a mirror image of the corresponding PDF from Subcase 1.1, reflected across the vertical axis at $\gamma = 0$.

Case 3: Coefficients of Opposite Sign $\alpha > 0, \beta < 0$. Let $\beta' = -\beta$.

Support: $[\beta, \alpha]$; Distribution: Parallelogram-shaped Distribution. This is the characteristic shape for coefficients of opposite signs; The support spans both negative and positive values. The PDF increases linearly, may maintain a constant value, and then decreases linearly, resulting in a distinctive parallelogram profile.

PDF:

$$f(\gamma = t) = \begin{cases} \frac{t - \beta}{\alpha\beta'} & t \in (\beta, \beta + \alpha) \\ \frac{1}{\alpha} & t \in (\beta + \alpha, 0) \\ \frac{\alpha - t}{\alpha\beta'} & t \in (0, \alpha) \end{cases} \quad (\text{A14})$$

Case 4: Coefficients of Opposite Sign $\alpha < 0, \beta > 0$. Let $\alpha' = -\alpha$.

Support: $[\alpha, \beta]$; Distribution: Parallelogram-shaped Distribution; Identical in form to Case 3, with the roles of the coefficients interchanged.

Appendix B. Let $r_1 \sim U(0, 1)$, $r_2 \sim U(0, 1)$, and r_1 and r_2 be independent. For any constants $\alpha, \beta > 0$, define the random variable $\gamma = r_1\alpha + r_2\beta$. Then the probability density function (PDF) of γ , denoted $f_\gamma(t)$, is symmetric about the point $m = \frac{\alpha+\beta}{2}$. That is

$$f_\gamma(m+t) = f_\gamma(m-t) \quad \text{for all } t \in \mathbb{R}.$$

The proof proceeds by directly examining the known piecewise form of the PDF and showing it possesses a specific reflection property.

The PDF of γ is derived from the convolution of the PDFs of $r_1\alpha + r_2\beta$. It is a piecewise linear function with support on the interval $[0, \alpha + \beta]$. Its exact form depends on the relationship between α and β :

1) For $t \in (0, \min(\alpha, \beta))$:

$$f_\gamma(t) = \frac{t}{\alpha\beta}.$$

2) For $t \in (\min(\alpha, \beta), \max(\alpha, \beta))$:

$$f_\gamma(t) = \frac{1}{\max(\alpha, \beta)}.$$

3) For $t \in (\max(\alpha, \beta), \alpha + \beta)$:

$$f_\gamma(t) = \frac{\alpha + \beta - t}{\alpha\beta}.$$

To prove symmetry about m , it is sufficient to prove that the PDF satisfies the following reflection property across the entire support:

$$f_\gamma(t) = f_\gamma(\alpha + \beta - t).$$

Observe that

$$m + t = \frac{\alpha + \beta}{2} + t, \quad m - t = \frac{\alpha + \beta}{2} - t,$$

and

$$\alpha + \beta - (m + t) = \alpha + \beta - \left(\frac{\alpha + \beta}{2} + t \right) = \frac{\alpha + \beta}{2} - t = m - t.$$

Therefore, if $f_\gamma(t) = f_\gamma(\alpha + \beta - t)$ holds, then by substituting $t = m + u$, we get

$$f_\gamma(m + u) = f_\gamma(\alpha + \beta - (m + u)) = f_\gamma(m - u),$$

which is the desired symmetry.

Now $f_\gamma(t) = f_\gamma(\alpha + \beta - t)$ for all t in the support by checking each piece of the function.

Case 1: $t \in (0, \min(\alpha, \beta))$

Here $f_\gamma(t) = \frac{t}{\alpha\beta}$. If t is in this interval, then $\alpha + \beta - t \in (\alpha + \beta - \min(\alpha, \beta), \alpha + \beta)$. Since $\alpha + \beta - \min(\alpha, \beta) = \max(\alpha, \beta)$, it follows that $\alpha + \beta - t \in (\max(\alpha, \beta), \alpha + \beta)$. In this interval, the PDF is $f_\gamma(s) = \frac{\alpha + \beta - s}{\alpha\beta}$. Substituting $s = \alpha + \beta - t$:

$$f_\gamma(\alpha + \beta - t) = \frac{\alpha + \beta - (\alpha + \beta - t)}{\alpha\beta} = \frac{t}{\alpha\beta} = f_\gamma(t).$$

Case 2: $t \in (\min(\alpha, \beta), \max(\alpha, \beta))$

Here $f_\gamma(t) = \frac{1}{\max(\alpha, \beta)}$. If t is in this interval, then $\alpha + \beta - t \in (\alpha + \beta - \max(\alpha, \beta), \alpha + \beta - \min(\alpha, \beta)) = (\min(\alpha, \beta), \max(\alpha, \beta))$. This is the same interval as Case 2, so the PDF here is also

$$f_\gamma(\alpha + \beta - t) = \frac{1}{\max(\alpha, \beta)} = f_\gamma(t).$$

Case 3: $t \in (\max(\alpha, \beta), \alpha + \beta)$

Here $f_\gamma(t) = \frac{\alpha + \beta - t}{\alpha\beta}$. If t is in this interval, then $\alpha + \beta - t \in (0, \alpha + \beta - \max(\alpha, \beta)) = (0, \min(\alpha, \beta))$. In this interval, the PDF is $f_\gamma(s) = \frac{s}{\alpha\beta}$. Substituting $s = \alpha + \beta - t$:

$$f_\gamma(\alpha + \beta - t) = \frac{\alpha + \beta - t}{\alpha\beta} = f_\gamma(t).$$

We have shown that $f_\gamma(t) = f_\gamma(\alpha + \beta - t)$ holds for all t in the support of γ . This directly implies that the probability density function is symmetric about the midpoint $m = \frac{\alpha + \beta}{2}$.

Author Biography



Mingyang Zhou received his Bachelor's degree from Hohai University, China, in 2018 and continued his academic journey there to pursue a Doctor of Philosophy (Ph.D.). His research primarily concentrates on intelligent algorithms and machine learning.



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