

## FALLING SHADOW REPRESENTATIONS OF FUZZY STRUCTURES IN HILBERT ALGEBRAS

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**ABSTRACT.** *Motivated by fuzzy set theory and falling shadow theory, we introduce the notions of falling subalgebras and falling ideals in Hilbert algebras via random sets. We show that every falling ideal is a falling subalgebra and present examples indicating that the converse does not hold. We further establish connections between classical fuzzy subalgebras (respectively, fuzzy ideals) and falling subalgebras (respectively, falling ideals), including a construction that represents each fuzzy subalgebra/ideal as the falling shadow of a suitable cut-cloud. In addition, we derive  $T_m$ -type inequalities for falling subalgebras and falling ideals, and we discuss  $I$ -fuzzy subalgebras and  $I$ -fuzzy ideals arising from falling inference relations.*

**Keywords:** Hilbert algebras, Falling shadow theory, Random sets, Falling subalgebras, Falling ideals

**1. Introduction.** In the study of unified treatments of uncertainty that combine probability and fuzzy set theory, Goodman [1] pointed out an equivalence between a fuzzy set and a class of random sets. Wang and Sanchez [2] introduced falling shadow theory, which connects probabilistic notions directly to the membership functions of fuzzy sets. Falling shadow representation theory provides a natural framework for selection based on joint degree distributions, and it has proved useful both for theoretical developments and for applications in fuzzy sets and fuzzy logic. A mathematical formulation of falling shadow theory was presented in [3]. Tan et al. [4, 5] developed a systematic approach to fuzzy inference relations and fuzzy set operations via falling shadows. In recent years, falling shadow methods have continued to generate algebraic characterizations and fuzzy-structural results in various logical algebras: Yang et al. [6] obtained characterizations of MV-algebras in terms of falling shadow theory, while Bandaru et al. [7, 8] investigated falling fuzzy filters and fuzzy implicative filters in BE-algebras through random-set representations. Related developments also appear in newer algebraic systems; for example, Jun et al. [9] applied falling shadow theory to UP-algebras, and Jun and Iampan [10] introduced the notion of falling UP-filters. Jun and Park [11] investigated falling fuzzy subalgebras and falling ideals in BCK/BCI-algebras. Yuan and Lee [12] interpreted fuzzy

subgroups (subrings, ideals) as falling shadows arising from suitable clouds of the corresponding crisp substructures.

Related fuzzy extensions in the setting of Hilbert algebras have also been studied recently. In particular, Iampan et al. [13] investigated intuitionistic hesitant fuzzy subalgebras and ideals of Hilbert algebras, providing a richer membership framework for these substructures. Moreover, Iampan et al. [14] developed several types of  $\mathcal{N}$ -fuzzy subalgebras in UP (BCC)-algebras, and Suayngam et al. [15] introduced Pythagorean fuzzy sets from the viewpoint of IUP-algebras, further illustrating the breadth of fuzzy algebraic models related to implication-based structures.

Despite these advances, falling-shadow-based representations for subalgebras and ideals in Hilbert algebras have not been developed as systematically as in other logical algebras. Hilbert algebras provide a basic implication-based framework in algebraic logic, where subalgebras and ideals reflect structural and proof-theoretic information via the induced order. However, most studies in algebraic settings have focused on falling-shadow-based fuzzy (implicative) filter formalisms in structures such as MV-algebras and BE-algebras, whereas analogous representations for subalgebras and ideals in Hilbert algebras have not been pursued to the same extent. This gap motivates the present work: we adapt random-set and cut-cloud constructions to the ideal conditions of Hilbert algebras and derive the corresponding falling and fuzzy characterizations.

Motivated by these developments, we study falling shadow constructions in the setting of Hilbert algebras. We introduce the notions of falling subalgebras and falling ideals via random sets, and we prove that every falling ideal is a falling subalgebra, while the converse fails in general. We then clarify the relationship between classical fuzzy subalgebras/ideals and falling subalgebras/ideals, including a representation of each fuzzy subalgebra/ideal as the falling shadow of a cut-cloud. Furthermore, we derive  $T_m$ -type inequalities associated with falling subalgebras and falling ideals and discuss  $I$ -fuzzy subalgebras and  $I$ -fuzzy ideals arising from falling inference relations.

**2. Preliminaries.** We summarize the notation and basic results used in the sequel. We recall Hilbert algebras and their subalgebras/ideals, review  $t$ -cuts of fuzzy sets, and outline the essentials of falling shadow theory via random sets. These tools will be used to define falling subalgebras and falling ideals and to derive the  $T_m$ -type and  $I$ -fuzzy conditions established later.

**Definition 2.1.** [16] *A Hilbert algebra is an algebraic structure  $X = (X, *, 1)$ , where  $X$  is a nonempty set,  $*$  is a binary operation on  $X$ , and  $1 \in X$  is a distinguished constant such that, for all  $x, y, z \in X$ , the following axioms hold:*

$$x * (y * x) = 1, \quad (1)$$

$$(x * (y * z)) * ((x * y) * (x * z)) = 1, \quad (2)$$

$$(x * y = 1 \text{ and } y * x = 1) \Rightarrow x = y. \quad (3)$$

In [17], the following identities were proved and will be used in the sequel.

**Lemma 2.1.** *Let  $X = (X, *, 1)$  be a Hilbert algebra. Then, for all  $x \in X$ ,*

$$(1) \ x * x = 1;$$

$$(2) \ 1 * x = x;$$

$$(3) \ x * 1 = 1.$$

In a Hilbert algebra  $X = (X, *, 1)$ , define a binary relation  $\leq$  on  $X$  by

$$x \leq y \iff x * y = 1 \quad (x, y \in X). \quad (4)$$

Then  $\leq$  is a partial order on  $X$ , and 1 is the greatest element with respect to  $\leq$ .

**Definition 2.2.** [18] Let  $X = (X, *, 1)$  be a Hilbert algebra. A nonempty subset  $S \subseteq X$  is called a subalgebra of  $X$  if it is closed under  $*$ ; that is,

$$x * y \in S \text{ for all } x, y \in S.$$

**Definition 2.3.** [19] Let  $X = (X, *, 1)$  be a Hilbert algebra. A nonempty subset  $S \subseteq X$  is called an ideal of  $X$  if it satisfies the following conditions.

- (1)  $1 \in S$ .
- (2) For all  $x, y \in X$ , if  $y \in S$ , then  $x * y \in S$ .
- (3) For all  $x, y_1, y_2 \in X$ , if  $y_1, y_2 \in S$ , then  $(y_1 * (y_2 * x)) * x \in S$ .

A fuzzy set [20] on a nonempty set  $X$  is a function  $\lambda : X \rightarrow [0, 1]$ , where  $[0, 1]$  denotes the closed unit interval of real numbers.

For  $t \in [0, 1]$ , the  $t$ -cut (or level set) of  $\lambda$  is defined by

$$\lambda_t = \{x \in X : \lambda(x) \geq t\}.$$

**Definition 2.4.** [21] Let  $X = (X, *, 1)$  be a Hilbert algebra. A fuzzy set  $\lambda : X \rightarrow [0, 1]$  is called a fuzzy subalgebra of  $X$  if, for all  $x, y \in X$ ,

$$\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}.$$

**Definition 2.5.** [22] Let  $X = (X, *, 1)$  be a Hilbert algebra. A fuzzy set  $\lambda : X \rightarrow [0, 1]$  is called a fuzzy ideal of  $X$  if, for all  $x, y, y_1, y_2 \in X$ , the following conditions hold:

- (1)  $\lambda(1) \geq \lambda(x)$ ;
- (2)  $\lambda(x * y) \geq \lambda(y)$ ;
- (3)  $\lambda((y_1 * (y_2 * x)) * x) \geq \min\{\lambda(y_1), \lambda(y_2)\}$ .

The next result is a standard level-cut (or  $t$ -cut) characterization for fuzzy substructures, which allows one to transfer algebraic properties between a fuzzy set and its nonempty level sets; see, e.g., [17, 20].

**Theorem 2.1.** Let  $\lambda$  be a fuzzy set on a Hilbert algebra  $X = (X, *, 1)$ . Then  $\lambda$  is a fuzzy subalgebra (respectively, a fuzzy ideal) of  $X$  if and only if, for every  $t \in [0, 1]$  with  $\lambda_t \neq \emptyset$ , the set  $\lambda_t$  is a subalgebra (respectively, an ideal) of  $X$ .

**3. Falling Subalgebras and Falling Ideals of Hilbert Algebras.** In this section, we introduce the main objects of the paper: falling subalgebras and falling ideals of a Hilbert algebra  $X = (X, *, 1)$ . Starting from a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ , we regard its falling shadow  $\tilde{\alpha} : X \rightarrow [0, 1]$  as a probabilistic representation of membership degrees. We call  $\tilde{\alpha}$  a falling subalgebra (respectively, a falling ideal) when the realizations  $\xi(\omega)$  form subalgebras (respectively, ideals) of  $X$ . We then develop basic properties and inclusion relations between these two notions, and we compare them with classical fuzzy subalgebras and fuzzy ideals via cut-cloud constructions. In addition, we derive  $T_m$ -type inequalities that naturally arise from probability bounds on intersections of events and discuss  $I$ -fuzzy variants motivated by falling inference relations.

We begin by recalling the basic notions of falling shadow theory; see [1, 2, 3, 4, 5] for further details. Let  $U$  be a universe of discourse, and let  $\mathcal{P}(U)$  denote the power set of  $U$ . For each  $u \in U$ , define

$$\ddot{u} = \{E \subseteq U : u \in E\},$$

and for each  $E \subseteq U$ , define

$$\ddot{E} = \{\ddot{u} : u \in E\}.$$

The pair  $(\mathcal{P}(U), \mathcal{B})$  is called a *hyper-measurable structure* on  $U$  if  $\mathcal{B}$  is a field of subsets of  $\mathcal{P}(U)$  and  $\ddot{U} \subseteq \mathcal{B}$ .

Let  $(\Omega, \mathcal{A}, P)$  be a probability space and let  $(\mathcal{P}(U), \mathcal{B})$  be a hyper-measurable structure on  $U$ . A *random set* on  $U$  is a mapping  $\xi : \Omega \rightarrow \mathcal{P}(U)$  that is  $\mathcal{A}$ - $\mathcal{B}$  measurable; that is,

$$\xi^{-1}(C) = \{\omega \in \Omega : \xi(\omega) \in C\} \in \mathcal{A} \text{ for all } C \in \mathcal{B}.$$

Let  $\xi$  be a random set on  $U$ . Define a function  $\tilde{\alpha} : U \rightarrow [0, 1]$  by

$$\tilde{\alpha}(u) = P(\{\omega \in \Omega : u \in \xi(\omega)\}) \quad (u \in U).$$

Then  $\tilde{\alpha}$  is a fuzzy set on  $U$ . We call  $\tilde{\alpha}$  the *falling shadow* of the random set  $\xi$ , and we call  $\xi$  a *cloud* of  $\tilde{\alpha}$ .

**Definition 3.1.** Let  $(\Omega, \mathcal{A}, P)$  be a probability space, and let  $\xi : \Omega \rightarrow \mathcal{P}(X)$  be a random set. Let  $\tilde{\alpha} : X \rightarrow [0, 1]$  be the falling shadow of  $\xi$ , defined by

$$\tilde{\alpha}(x) = P(\{\omega \in \Omega : x \in \xi(\omega)\}) \quad (x \in X).$$

If  $\xi(\omega)$  is an ideal (respectively, a subalgebra) of  $X$  for every  $\omega \in \Omega$ , then  $\tilde{\alpha}$  is called a falling ideal (respectively, a falling subalgebra) of  $X$ .

**Proposition 3.1.** Let  $\tilde{\alpha}$  be a falling subalgebra of  $X$  induced by a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . Then, for all  $x \in X$ ,

$$\Omega_{\xi}(x) \subseteq \Omega_{\xi}(1).$$

**Proof:** Let  $x \in X$  and let  $\omega \in \Omega_{\xi}(x)$ . Then  $x \in \xi(\omega)$ . Since  $\tilde{\alpha}$  is a falling subalgebra,  $\xi(\omega)$  is a subalgebra of  $X$ ; hence,  $1 \in \xi(\omega)$ . Therefore,  $\omega \in \Omega_{\xi}(1)$ . This shows that  $\Omega_{\xi}(x) \subseteq \Omega_{\xi}(1)$  for all  $x \in X$ .  $\square$

**Proposition 3.2.** Let  $\tilde{\alpha}$  be a falling subalgebra of  $X$  induced by a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . Then, for all  $x \in X$ ,

$$\Omega_{\xi}(x) \subseteq \Omega_{\xi}(1 * x).$$

**Proof:** Let  $x \in X$  and let  $\omega \in \Omega_{\xi}(x)$ . Then  $x \in \xi(\omega)$ . Since  $\tilde{\alpha}$  is a falling subalgebra,  $\xi(\omega)$  is a subalgebra of  $X$ ; hence, it is closed under  $*$  and contains 1. Therefore,  $1 * x \in \xi(\omega)$ . This implies  $\omega \in \Omega_{\xi}(1 * x)$ , and consequently  $\Omega_{\xi}(x) \subseteq \Omega_{\xi}(1 * x)$ .  $\square$

**Theorem 3.1.** Every falling ideal of  $X$  is a falling subalgebra of  $X$ .

**Proof:** Let  $\tilde{\alpha}$  be a falling ideal of  $X$  induced by a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . Then  $\xi(\omega)$  is an ideal of  $X$  for every  $\omega \in \Omega$ . Since every ideal of a Hilbert algebra is, in particular, a subalgebra (i.e., it is closed under  $*$ ), it follows that  $\xi(\omega)$  is a subalgebra of  $X$  for every  $\omega \in \Omega$ . Hence,  $\tilde{\alpha}$  is a falling subalgebra of  $X$ .  $\square$

The following example shows that the converse of Theorem 3.1 is not true in a Hilbert algebra.

**Example 3.1.** Let  $X = \{1, x, y, z, 0\}$  be a Hilbert algebra with the binary operation  $*$  given in Table 1.

TABLE 1. Cayley table for the binary operation  $*$

| $*$ | 1 | $x$ | $y$ | $z$ | 0   |
|-----|---|-----|-----|-----|-----|
| 1   | 1 | $x$ | $y$ | $z$ | 0   |
| $x$ | 1 | 1   | $y$ | $z$ | 0   |
| $y$ | 1 | $x$ | 1   | $z$ | $z$ |
| $z$ | 1 | 1   | $y$ | 1   | $y$ |
| 0   | 1 | 1   | 1   | 1   | 1   |

*Note.* From Table 1, we record the entries used below:  $z * x = 1$ ,  $1 * 1 = 1$ , and  $1 * x = x$ . Hence,  $(1 * (z * x)) * x = (1 * 1) * x = 1 * x = x$ .

Let  $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ , where  $m$  denotes the Lebesgue measure, and define a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$  by

$$\xi(t) = \begin{cases} \{1, x\}, & t \in [0, 0.8), \\ \{1, z\}, & t \in [0.8, 1]. \end{cases}$$

Then  $\xi(t)$  is a subalgebra of  $X$  for every  $t \in [0, 1]$ . Hence, the falling shadow  $\tilde{\alpha} : X \rightarrow [0, 1]$ , defined by  $\tilde{\alpha}(a) = m(\{t \in [0, 1] : a \in \xi(t)\})$ , is a falling subalgebra of  $X$ . Moreover,

$$\tilde{\alpha}(1) = 1, \quad \tilde{\alpha}(x) = 0.8, \quad \tilde{\alpha}(z) = 0.2, \quad \tilde{\alpha}(y) = \tilde{\alpha}(0) = 0.$$

However,  $\xi(t) = \{1, z\}$  is not an ideal of  $X$  (for any  $t \in [0.8, 1]$ ). Indeed, taking  $y_1 = 1$ ,  $y_2 = z \in \xi(t)$  and  $x = x \in X$ , we have

$$(1 * (z * x)) * x = x \notin \{1, z\} = \xi(t),$$

so condition (3) in the definition of an ideal fails. Therefore,  $\tilde{\alpha}$  is not a falling ideal of  $X$ .

**Proposition 3.3.** Let  $\tilde{\alpha}$  be a falling ideal of  $X$  induced by a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . Then the following statements hold:

- (1) For all  $x, y \in X$ ,  $\Omega_\xi(y) \subseteq \Omega_\xi((y * x) * x)$ .
- (2) For all  $x, y \in X$ , if  $x \leq y$ , then  $\Omega_\xi(x) \subseteq \Omega_\xi(y)$ .

**Proof:**

(1) Let  $x, y \in X$  and let  $\omega \in \Omega_\xi(y)$ . Then  $y \in \xi(\omega)$ . Since  $\tilde{\alpha}$  is a falling ideal,  $\xi(\omega)$  is an ideal of  $X$ . In particular,  $1 \in \xi(\omega)$ , and by condition (3) in the definition of an ideal (with  $y_1 = y$ ,  $y_2 = 1$ , and  $x = x$ ), we obtain

$$(y * (1 * x)) * x \in \xi(\omega).$$

Using  $1 * x = x$ , this yields  $(y * x) * x \in \xi(\omega)$ ; hence,  $\omega \in \Omega_\xi((y * x) * x)$ . Therefore,  $\Omega_\xi(y) \subseteq \Omega_\xi((y * x) * x)$ .

(2) Let  $x, y \in X$  with  $x \leq y$ , and let  $\omega \in \Omega_\xi(x)$ . Then  $x \in \xi(\omega)$ . By the definition of the order  $\leq$  (see Equation (4)), we have  $x * y = 1$ . Since  $\xi(\omega)$  is an ideal of  $X$ , condition (3) in Definition 2.3 links the order structure with ideal membership by allowing one to derive larger elements (with respect to  $\leq$ ) from elements already in the ideal. Indeed, because  $1 \in \xi(\omega)$  and  $x \in \xi(\omega)$ , applying condition (3) with  $y_1 = 1$ ,  $y_2 = x$ , and  $x = y$  yields

$$(1 * (x * y)) * y \in \xi(\omega).$$

Using  $x * y = 1$  and  $1 * 1 = 1$ , we obtain  $(1 * (x * y)) * y = (1 * 1) * y = 1 * y = y$ ; hence,  $y \in \xi(\omega)$ . Therefore,  $\omega \in \Omega_\xi(y)$ , and thus  $\Omega_\xi(x) \subseteq \Omega_\xi(y)$ .  $\square$

**Remark 3.1.** Let  $X = (X, *, 1)$  be a Hilbert algebra and let  $(\Omega, \mathcal{A}, P)$  be a probability space. Set

$$F(X) = \{f : \Omega \rightarrow X\}.$$

Define a binary operation  $\odot$  on  $F(X)$  pointwise by

$$(f \odot g)(\omega) = f(\omega) * g(\omega) \quad (\omega \in \Omega),$$

for all  $f, g \in F(X)$ . Let  $\theta \in F(X)$  be the constant map  $\theta(\omega) = 1$  for all  $\omega \in \Omega$ . Then  $(F(X), \odot, \theta)$  is a Hilbert algebra.

Let  $A \subseteq X$  and  $f \in F(X)$ . Define  $A_f = f^{-1}(A)$ . Then  $A_f \in \mathcal{A}$ .

**Theorem 3.2.** *Let  $X = (X, *, 1)$  be a Hilbert algebra. Let  $F(X)$ ,  $\odot$ ,  $\theta$ , and  $\xi(\omega)$  be defined as in Remark 3.1. If  $A$  is a subalgebra (respectively, an ideal) of  $X$ , then  $\xi(\omega)$  is a subalgebra (respectively, an ideal) of  $F(X)$  for every  $\omega \in \Omega$ .*

**Proof:** Fix  $\omega \in \Omega$ .

*Subalgebra case.* Assume that  $A$  is a subalgebra of  $X$ . Let  $f, g \in \xi(\omega)$ . Then  $f(\omega), g(\omega) \in A$ . Since  $A$  is closed under  $*$ , we have  $f(\omega) * g(\omega) \in A$ ; hence,  $(f \odot g)(\omega) = f(\omega) * g(\omega) \in A$ . Therefore,  $f \odot g \in \xi(\omega)$ , so  $\xi(\omega)$  is a subalgebra of  $F(X)$ .

*Ideal case.* Assume that  $A$  is an ideal of  $X$ . Then  $1 \in A$ , so  $\theta(\omega) = 1 \in A$ ; hence,  $\theta \in \xi(\omega)$ . Let  $f, g \in F(X)$  with  $g \in \xi(\omega)$ . Then  $g(\omega) \in A$ , and by condition (2) in the definition of an ideal,  $f(\omega) * g(\omega) \in A$ . Thus,  $f \odot g \in \xi(\omega)$ .

Finally, let  $f, g, h \in F(X)$  with  $f, g \in \xi(\omega)$ . Then  $f(\omega), g(\omega) \in A$ , and by condition (3) we obtain  $(f(\omega) * (g(\omega) * h(\omega))) * h(\omega) \in A$ . Equivalently,  $((f \odot (g \odot h)) \odot h)(\omega) \in A$ , so  $(f \odot (g \odot h)) \odot h \in \xi(\omega)$ . Hence,  $\xi(\omega)$  is an ideal of  $F(X)$ .  $\square$

Let  $A \subseteq X$  and define  $\xi : \Omega \rightarrow \mathcal{P}(F(X))$  by

$$\xi(\omega) = \{f \in F(X) : f(\omega) \in A\} \quad (\omega \in \Omega).$$

Let  $\mathcal{B}$  be the field on  $\mathcal{P}(F(X))$  generated by  $\{\ddot{f} : f \in F(X)\}$ . For each  $f \in F(X)$ , we have  $\xi^{-1}(\ddot{f}) = \{\omega \in \Omega : f \in \xi(\omega)\}$ , and hence

$$\xi^{-1}(\ddot{f}) = \{\omega \in \Omega : f(\omega) \in A\} = A_f \in \mathcal{A}.$$

Therefore,  $\xi$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable, and hence  $\xi$  is a random set on  $F(X)$ .

Its falling shadow  $\tilde{\alpha} : F(X) \rightarrow [0, 1]$  is defined by

$$\tilde{\alpha}(f) = P(\{\omega \in \Omega : f(\omega) \in A\}) \quad (f \in F(X)).$$

Consequently, if  $A$  is a subalgebra (respectively, an ideal) of  $X$ , then  $\tilde{\alpha}$  is a falling subalgebra (respectively, a falling ideal) of  $F(X)$ .

**Theorem 3.3.** *Every fuzzy subalgebra (respectively, fuzzy ideal)  $\lambda$  of  $X$  is a falling subalgebra (respectively, a falling ideal) of  $X$ .*

**Proof:** Let  $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ , where  $\mathcal{A}$  is the Borel  $\sigma$ -algebra on  $[0, 1]$  and  $m$  is the Lebesgue measure. Let  $\lambda : X \rightarrow [0, 1]$  be a fuzzy subalgebra (respectively, a fuzzy ideal) of  $X$ . For each  $t \in [0, 1]$ , consider the  $t$ -cut  $\lambda_t = \{x \in X : \lambda(x) \geq t\}$ . By Theorem 2.1 (the cut-set characterization),  $\lambda_t$  is a subalgebra (respectively, an ideal) of  $X$  whenever  $\lambda_t \neq \emptyset$ .

Define  $\xi : \Omega \rightarrow \mathcal{P}(X)$  by  $\xi(t) = \lambda_t$  for  $t \in [0, 1]$ . Then  $\xi(t)$  is a subalgebra (respectively, an ideal) of  $X$  for every  $t \in [0, 1]$ . Moreover, for each  $x \in X$ ,

$$\{t \in [0, 1] : x \in \xi(t)\} = \{t \in [0, 1] : x \in \lambda_t\} = \{t \in [0, 1] : t \leq \lambda(x)\} = [0, \lambda(x)],$$

which is a Borel set. Hence,  $\xi$  is a random set on  $X$ , and its falling shadow  $\tilde{\alpha}$  satisfies

$$\tilde{\alpha}(x) = P(\{t \in [0, 1] : x \in \xi(t)\}) = m([0, \lambda(x)]) = \lambda(x) \quad (x \in X).$$

Therefore,  $\lambda$  coincides with the falling shadow of  $\xi$ , and thus  $\lambda$  is a falling subalgebra (respectively, a falling ideal) of  $X$ .  $\square$

The converse of Theorem 3.3 need not hold, as shown by the following example.

**Example 3.2.** *Let  $X = \{1, x, y, z, 0\}$  be the Hilbert algebra whose operation  $*$  is given in Table 1. Let  $\Omega = \{\omega_1, \omega_2, \omega_3\}$ ,  $\mathcal{A} = \mathcal{P}(\Omega)$ , and define a probability measure  $P$  on  $\Omega$  by*

$$P(\{\omega_1\}) = 0.4, \quad P(\{\omega_2\}) = 0.4, \quad P(\{\omega_3\}) = 0.2.$$

Define a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$  by

$$\xi(\omega_1) = \{1, x\}, \quad \xi(\omega_2) = \{1, y\}, \quad \xi(\omega_3) = X.$$

Each  $\xi(\omega_i)$  is a subalgebra of  $X$ ; hence, the falling shadow  $\tilde{\alpha} : X \rightarrow [0, 1]$ ,

$$\tilde{\alpha}(a) = P(\{\omega \in \Omega : a \in \xi(\omega)\}) \quad (a \in X),$$

is a falling subalgebra of  $X$ . In particular,

$$\tilde{\alpha}(1) = 1, \quad \tilde{\alpha}(x) = 0.6, \quad \tilde{\alpha}(y) = 0.6, \quad \tilde{\alpha}(z) = 0.2, \quad \tilde{\alpha}(0) = 0.2.$$

Note. From Table 1, we have  $x * y = z$ , which is the key entry used to obtain  $\tilde{\alpha}(x * y) = \tilde{\alpha}(z)$  in the following inequality. Hence,

$$\tilde{\alpha}(x * y) = \tilde{\alpha}(z) = 0.2 < 0.6 = \min\{\tilde{\alpha}(x), \tilde{\alpha}(y)\}.$$

Therefore,  $\tilde{\alpha}$  is not a fuzzy subalgebra of  $X$ .

Let  $(\Omega, \mathcal{A}, P)$  be a probability space, and let  $\xi : \Omega \rightarrow \mathcal{P}(X)$  be a random set with falling shadow  $\tilde{\alpha}$ . For each  $x \in X$ , define

$$\Omega_\xi(x) = \{\omega \in \Omega : x \in \xi(\omega)\}.$$

Then  $\Omega_\xi(x) \in \mathcal{A}$  for every  $x \in X$ .

**Proposition 3.4.** Let  $\tilde{\alpha}$  be the falling shadow of a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . If  $\tilde{\alpha}$  is a falling subalgebra of  $X$ , then for all  $x, y \in X$ ,

$$\Omega_\xi(x) \cap \Omega_\xi(y) \subseteq \Omega_\xi(x * y). \quad (5)$$

**Proof:** Let  $x, y \in X$  and let  $\omega \in \Omega_\xi(x) \cap \Omega_\xi(y)$ . Then  $x \in \xi(\omega)$  and  $y \in \xi(\omega)$ . Since  $\tilde{\alpha}$  is a falling subalgebra,  $\xi(\omega)$  is a subalgebra of  $X$ ; hence, it is closed under  $*$ , and therefore  $x * y \in \xi(\omega)$ . Thus,  $\omega \in \Omega_\xi(x * y)$ . This proves Equation (5).  $\square$

**Proposition 3.5.** Let  $\tilde{\alpha}$  be the falling shadow of a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . If  $\tilde{\alpha}$  is a falling ideal of  $X$ , then for all  $x, y_1, y_2 \in X$ ,

$$\Omega_\xi(y_1) \cap \Omega_\xi(y_2) \subseteq \Omega_\xi((y_1 * (y_2 * x)) * x). \quad (6)$$

**Proof:** Let  $x, y_1, y_2 \in X$  and let  $\omega \in \Omega_\xi(y_1) \cap \Omega_\xi(y_2)$ . Then  $y_1 \in \xi(\omega)$  and  $y_2 \in \xi(\omega)$ . Since  $\tilde{\alpha}$  is a falling ideal,  $\xi(\omega)$  is an ideal of  $X$ ; hence, by condition (3) in the definition of an ideal,

$$(y_1 * (y_2 * x)) * x \in \xi(\omega).$$

Therefore,  $\omega \in \Omega_\xi((y_1 * (y_2 * x)) * x)$ , which proves Equation (6).  $\square$

For  $s, t \in [0, 1]$ , define

$$T_m(s, t) = \max\{0, s + t - 1\}.$$

The operator  $T_m$  is the Łukasiewicz  $t$ -norm, which is a standard fuzzy conjunction. Heuristically, if  $s$  and  $t$  represent membership degrees (or truth degrees) of two fuzzy properties, then  $T_m(s, t)$  measures the degree to which both properties hold simultaneously. In particular,  $T_m(s, t) = 0$  when  $s + t \leq 1$ , reflecting insufficient combined support for joint satisfaction, while  $T_m(s, t)$  increases linearly to 1 as  $s$  and  $t$  approach 1. In our framework,  $T_m$  arises naturally from the probabilistic estimate  $P(A \cap B) \geq P(A) + P(B) - 1$ , which links the falling shadow of an intersection event to the falling shadows of its components.

**Theorem 3.4.** Let  $\tilde{\alpha}$  be the falling shadow of a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . If  $\tilde{\alpha}$  is a falling subalgebra of  $X$ , then for all  $x, y \in X$ ,

$$\tilde{\alpha}(x * y) \geq T_m(\tilde{\alpha}(x), \tilde{\alpha}(y)).$$

**Proof:** Let  $x, y \in X$ . By Proposition 3.4,

$$\Omega_\xi(x) \cap \Omega_\xi(y) \subseteq \Omega_\xi(x * y).$$

Hence,

$$\tilde{\alpha}(x * y) = P(\Omega_\xi(x * y)) \geq P(\Omega_\xi(x) \cap \Omega_\xi(y)).$$

Using the inequality  $P(A \cap B) \geq P(A) + P(B) - 1$  (for events  $A, B \in \mathcal{A}$ ), we obtain

$$P(\Omega_\xi(x) \cap \Omega_\xi(y)) \geq P(\Omega_\xi(x)) + P(\Omega_\xi(y)) - 1 = \tilde{\alpha}(x) + \tilde{\alpha}(y) - 1.$$

Therefore,

$$\tilde{\alpha}(x * y) \geq \max\{0, \tilde{\alpha}(x) + \tilde{\alpha}(y) - 1\} = T_m(\tilde{\alpha}(x), \tilde{\alpha}(y)). \quad \square$$

**Theorem 3.5.** Let  $\tilde{\alpha}$  be the falling shadow of a random set  $\xi : \Omega \rightarrow \mathcal{P}(X)$ . If  $\tilde{\alpha}$  is a falling ideal of  $X$ , then the following statements hold.

- (1) For all  $x \in X$ ,  $\tilde{\alpha}(1) \geq \tilde{\alpha}(x)$ .
- (2) For all  $x, y \in X$ ,  $\tilde{\alpha}(x * y) \geq \tilde{\alpha}(y)$ .
- (3) For all  $x, y_1, y_2 \in X$ ,

$$\tilde{\alpha}((y_1 * (y_2 * x)) * x) \geq T_m(\tilde{\alpha}(y_1), \tilde{\alpha}(y_2)),$$

where  $T_m(s, t) = \max\{0, s + t - 1\}$  for  $s, t \in [0, 1]$ .

**Proof:**

- (1) By Proposition 3.1,  $\Omega_\xi(x) \subseteq \Omega_\xi(1)$  for all  $x \in X$ . Hence,

$$\tilde{\alpha}(1) = P(\Omega_\xi(1)) \geq P(\Omega_\xi(x)) = \tilde{\alpha}(x).$$

- (2) By Proposition 3.3,  $\Omega_\xi(y) \subseteq \Omega_\xi(x * y)$  for all  $x, y \in X$ . Therefore,

$$\tilde{\alpha}(x * y) = P(\Omega_\xi(x * y)) \geq P(\Omega_\xi(y)) = \tilde{\alpha}(y).$$

- (3) By Proposition 3.5,

$$\Omega_\xi(y_1) \cap \Omega_\xi(y_2) \subseteq \Omega_\xi((y_1 * (y_2 * x)) * x) \quad (x, y_1, y_2 \in X).$$

Hence,

$$\tilde{\alpha}((y_1 * (y_2 * x)) * x) = P(\Omega_\xi((y_1 * (y_2 * x)) * x)) \geq P(\Omega_\xi(y_1) \cap \Omega_\xi(y_2)).$$

Using  $P(A \cap B) \geq P(A) + P(B) - 1$ , we obtain

$$P(\Omega_\xi(y_1) \cap \Omega_\xi(y_2)) \geq P(\Omega_\xi(y_1)) + P(\Omega_\xi(y_2)) - 1 = \tilde{\alpha}(y_1) + \tilde{\alpha}(y_2) - 1.$$

Therefore,

$$\tilde{\alpha}((y_1 * (y_2 * x)) * x) \geq \max\{0, \tilde{\alpha}(y_1) + \tilde{\alpha}(y_2) - 1\} = T_m(\tilde{\alpha}(y_1), \tilde{\alpha}(y_2)). \quad \square$$

Theorem 3.5 shows that every falling ideal of  $X$  is a  $T_m$ -fuzzy ideal of  $X$ . However, the converse need not hold, as illustrated by the following example.

**Example 3.3.** Let  $X = \{1, x, y, z, 0\}$  be the Hilbert algebra whose operation  $*$  is given in Table 1. Let  $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ , where  $\mathcal{A}$  is the Borel  $\sigma$ -algebra on  $[0, 1]$  and  $m$  is the Lebesgue measure. Define  $\xi : \Omega \rightarrow \mathcal{P}(X)$  by

$$\xi(t) = \begin{cases} \{1, x\}, & t \in [0, 0.6), \\ \{y, z, 0\}, & t \in [0.6, 1]. \end{cases}$$

Then  $\xi(t)$  is not an ideal of  $X$  for  $t \in [0.6, 1]$ . Hence, the falling shadow  $\tilde{\alpha}$  of  $\xi$  is not a falling ideal of  $X$ . Moreover, for  $a \in X$ ,

$$\tilde{\alpha}(a) = m(\{t \in [0, 1] : a \in \xi(t)\}),$$

and therefore

$$\tilde{\alpha}(1) = \tilde{\alpha}(x) = 0.6, \quad \tilde{\alpha}(y) = \tilde{\alpha}(z) = \tilde{\alpha}(0) = 0.4.$$

On the other hand, one can verify directly from Table 1 that  $\tilde{\alpha}$  satisfies conditions (1)-(3) in Theorem 3.5; hence,  $\tilde{\alpha}$  is a  $T_m$ -fuzzy ideal of  $X$ .

Based on the properties of falling subalgebras and falling ideals established above, we further extend the falling shadow framework to a fuzzy reasoning scenario by considering falling inference relations in Hilbert algebras. In 1993, Tan et al. [4] proposed a method for defining fuzzy inference relations via falling shadow theory. Let  $A$  and  $B$  be fuzzy sets on universes  $U$  and  $V$ , respectively. For  $t \in [0, 1]$ , denote the  $t$ -cuts by

$$A_t = \{u \in U : A(u) \geq t\}, \quad B_t = \{v \in V : B(v) \geq t\}.$$

Let  $\xi$  and  $\zeta$  be cut-clouds of  $A$  and  $B$ , respectively, so that  $\xi(t) = A_t$  and  $\zeta(t) = B_t$  on  $[0, 1]$ .

Originally,  $\xi$  and  $\zeta$  may be defined on two probability spaces  $([0, 1], \mathcal{B}_1, m_1)$  and  $([0, 1], \mathcal{B}_2, m_2)$ , where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are the Borel  $\sigma$ -algebras on  $[0, 1]$  and  $m_1, m_2$  are Lebesgue measures. Following [4], we regard  $\xi$  and  $\zeta$  on a unified probability space  $([0, 1]^2, \mathcal{B}, P)$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[0, 1]^2$  and  $P$  is a joint probability measure, by setting

$$\xi(t, s) = A_t, \quad \zeta(t, s) = B_s \quad ((t, s) \in [0, 1]^2).$$

Thus, for each  $(t, s) \in [0, 1]^2$ ,  $\xi(t, s)$  and  $\zeta(t, s)$  are crisp sets in  $U$  and  $V$ , respectively.

For crisp sets  $A_t \subseteq U$  and  $B_s \subseteq V$ , the usual implication relation  $A_t \rightarrow B_s$  is represented by

$$I_{A_t \rightarrow B_s} = (A_t \times B_s) \cup (A_t^c \times V) \subseteq U \times V.$$

This yields a random set  $(t, s) \mapsto I_{A_t \rightarrow B_s}$  on  $U \times V$ . Identifying its falling shadow, the fuzzy inference relation  $I_{A \rightarrow B}$  of the implication  $A \rightarrow B$  is defined by

$$I_{A \rightarrow B}(u, v) = P(\{(t, s) \in [0, 1]^2 : (u, v) \in I_{A_t \rightarrow B_s}\}) \quad ((u, v) \in U \times V).$$

Since  $P$  is a joint probability measure on  $[0, 1]^2$ , different choices of  $P$  generally lead to different explicit formulas for  $I_{A \rightarrow B}$  [4]. In particular, if  $P$  is the diagonal distribution concentrated on  $\{(t, t) : t \in [0, 1]\}$ , then

$$I_{A \rightarrow B}(u, v) = \min\{1 - A(u) + B(v), 1\}.$$

We now introduce the notions of  $I$ -fuzzy subalgebras and  $I$ -fuzzy ideals in Hilbert algebras.

**Definition 3.2.** Let  $I : [0, 1]^2 \rightarrow [0, 1]$  be a falling implication operator and let  $t \in (0, 1]$ . A fuzzy set  $\lambda : X \rightarrow [0, 1]$  is called an  $I$ -fuzzy subalgebra of  $X$  with respect to  $t$  if, for all  $x, y \in X$ ,

$$I(\min\{\lambda(x), \lambda(y)\}, \lambda(x * y)) \geq t. \quad (7)$$

**Definition 3.3.** Let  $I : [0, 1]^2 \rightarrow [0, 1]$  be a falling implication operator and let  $t \in (0, 1]$ . A fuzzy set  $\lambda : X \rightarrow [0, 1]$  is called an  $I$ -fuzzy ideal of  $X$  with respect to  $t$  if, for all  $x, y, y_1, y_2 \in X$ , the following conditions hold.

- (1)  $I(\lambda(x), \lambda(1)) \geq t$ .
- (2)  $I(\lambda(x * y), \lambda(y)) \geq t$ .
- (3)  $I(\min\{\lambda(y_1), \lambda(y_2)\}, \lambda((y_1 * (y_2 * x)) * x)) \geq t$ .

These definitions generalize the classical fuzzy subalgebra/ideal conditions by using implication-based thresholds  $I(\cdot, \cdot) \geq t$  instead of min-based inequalities. The parameters  $I$  and  $t$  encode the underlying reasoning semantics and the required degree of satisfaction. In particular, under the diagonal distribution and  $t = 1$ , the  $I$ -fuzzy notions coincide with the usual fuzzy subalgebra/ideal notions.

If  $P$  is the diagonal distribution, then the notion of an  $I$ -fuzzy subalgebra (respectively, an  $I$ -fuzzy ideal) with respect to  $t = 1$  coincides with the usual notion of a fuzzy subalgebra (respectively, a fuzzy ideal).

**Theorem 3.6.** *Let  $\lambda$  be a fuzzy set on  $X$  and let  $t = 0.5$ . If  $P$  is the diagonal distribution, then the following statements are equivalent.*

(1)  $\lambda$  is an  $I$ -fuzzy subalgebra of  $X$  with respect to  $t = 0.5$ .

(2) For all  $x, y \in X$ , either  $\min\{\lambda(x), \lambda(y)\} \leq \lambda(x * y)$ , or

$$0 < \min\{\lambda(x), \lambda(y)\} - \lambda(x * y) \leq 0.5.$$

**Proof:**

(1) $\Rightarrow$ (2). Assume that  $\lambda$  is an  $I$ -fuzzy subalgebra of  $X$  with respect to  $t = 0.5$ . Under the diagonal distribution, the implication is the Łukasiewicz implication

$$I(a, b) = \min\{1 - a + b, 1\} \quad (a, b \in [0, 1]).$$

Let  $x, y \in X$  and set  $a = \min\{\lambda(x), \lambda(y)\}$  and  $b = \lambda(x * y)$ . Then

$$0.5 \leq I(a, b) = \min\{1 - a + b, 1\}.$$

If  $a \leq b$ , then (2) holds. If  $a > b$ , then  $I(a, b) = 1 - a + b$ , and hence

$$0.5 \leq 1 - a + b \iff a - b \leq 0.5.$$

Since  $a > b$ , we also have  $0 < a - b$ . Therefore,  $0 < a - b \leq 0.5$ , which is exactly (2).

(2) $\Rightarrow$ (1). Assume that (2) holds. Let  $x, y \in X$  and set  $a = \min\{\lambda(x), \lambda(y)\}$ ,  $b = \lambda(x * y)$ . If  $a \leq b$ , then  $I(a, b) = 1 \geq 0.5$ . If  $0 < a - b \leq 0.5$ , then  $I(a, b) = 1 - a + b \geq 0.5$ . In either case,

$$I(\min\{\lambda(x), \lambda(y)\}, \lambda(x * y)) \geq 0.5,$$

so  $\lambda$  is an  $I$ -fuzzy subalgebra of  $X$  with respect to  $t = 0.5$ .  $\square$

**Theorem 3.7.** *Let  $\lambda$  be a fuzzy set on  $X$  and let  $t = 0.5$ . If  $P$  is the diagonal distribution, then the following statements are equivalent.*

(1)  $\lambda$  is an  $I$ -fuzzy ideal of  $X$  with respect to  $t = 0.5$ .

(2) The following conditions hold:

$$(\forall x \in X) (\lambda(x) \leq \lambda(1) \text{ or } 0 < \lambda(x) - \lambda(1) \leq 0.5), \quad (8)$$

$$(\forall x, y \in X) (\lambda(x * y) \leq \lambda(y) \text{ or } 0 < \lambda(x * y) - \lambda(y) \leq 0.5), \quad (9)$$

$$\begin{aligned} &(\forall x, y_1, y_2 \in X) (\min\{\lambda(y_1), \lambda(y_2)\} \leq \lambda((y_1 * (y_2 * x)) * x) \\ &\text{or } 0 < \min\{\lambda(y_1), \lambda(y_2)\} - \lambda((y_1 * (y_2 * x)) * x) \leq 0.5). \end{aligned} \quad (10)$$

**Proof:** Under the diagonal distribution, the implication is the Łukasiewicz implication

$$I(a, b) = \min\{1 - a + b, 1\} \quad (a, b \in [0, 1]).$$

(1) $\Rightarrow$ (2). Assume that  $\lambda$  is an  $I$ -fuzzy ideal of  $X$  with respect to  $t = 0.5$ .

*Condition (8).* Fix  $x \in X$ . From Definition 3.3 (1),

$$0.5 \leq I(\lambda(x), \lambda(1)) = \min\{1 - \lambda(x) + \lambda(1), 1\}.$$

If  $\lambda(x) \leq \lambda(1)$ , then Equation (8) holds. If  $\lambda(x) > \lambda(1)$ , then  $I(\lambda(x), \lambda(1)) = 1 - \lambda(x) + \lambda(1) \geq 0.5$ , so  $0 < \lambda(x) - \lambda(1) \leq 0.5$ . Hence, Equation (8) holds.

*Condition (9).* Fix  $x, y \in X$ . From Definition 3.3 (2),

$$0.5 \leq I(\lambda(x * y), \lambda(y)) = \min\{1 - \lambda(x * y) + \lambda(y), 1\}.$$

Arguing as above with  $a = \lambda(x * y)$  and  $b = \lambda(y)$ , we obtain Equation (9).

*Condition (10).* Fix  $x, y_1, y_2 \in X$ . From Definition 3.3 (3),

$$0.5 \leq I(\min\{\lambda(y_1), \lambda(y_2)\}, \lambda((y_1 * (y_2 * x)) * x)).$$

Let  $a = \min\{\lambda(y_1), \lambda(y_2)\}$  and  $b = \lambda((y_1 * (y_2 * x)) * x)$ . Then  $0.5 \leq \min\{1 - a + b, 1\}$ , and the same two-case argument ( $a \leq b$  or  $a > b$ ) yields Equation (10).

(2) $\Rightarrow$ (1). Assume that Equations (8)-(10) hold.

*Verification of (1).* Fix  $x \in X$  and set  $a = \lambda(x)$ ,  $b = \lambda(1)$ . If  $a \leq b$ , then  $I(a, b) = 1 \geq 0.5$ . If  $0 < a - b \leq 0.5$ , then  $I(a, b) = 1 - a + b \geq 0.5$ . Thus,  $I(\lambda(x), \lambda(1)) \geq 0.5$ .

*Verification of (2).* Fix  $x, y \in X$  and set  $a = \lambda(x * y)$ ,  $b = \lambda(y)$ . By Equation (9), either  $a \leq b$  or  $0 < a - b \leq 0.5$ , and in both cases  $I(a, b) \geq 0.5$ . Hence,  $I(\lambda(x * y), \lambda(y)) \geq 0.5$ .

*Verification of (3).* Fix  $x, y_1, y_2 \in X$  and set  $a = \min\{\lambda(y_1), \lambda(y_2)\}$  and  $b = \lambda((y_1 * (y_2 * x)) * x)$ . By Equation (10), either  $a \leq b$  or  $0 < a - b \leq 0.5$ , so  $I(a, b) \geq 0.5$ . Therefore,  $I(\min\{\lambda(y_1), \lambda(y_2)\}, \lambda((y_1 * (y_2 * x)) * x)) \geq 0.5$ .

Combining these three verifications,  $\lambda$  is an  $I$ -fuzzy ideal of  $X$  with respect to  $t = 0.5$ .  $\square$

**4. Conclusion.** In this paper, we developed falling shadow constructions in the setting of Hilbert algebras. Based on random sets and their falling shadows, we introduced the notions of falling subalgebras and falling ideals and examined their basic properties. In particular, we proved that every falling ideal is a falling subalgebra, and we provided examples showing that the converse does not hold in general.

We also clarified the relationship between classical fuzzy subalgebras/ideals and falling subalgebras/ideals. In one direction, we constructed a cut-cloud whose falling shadow coincides with a given fuzzy subalgebra (respectively, fuzzy ideal), showing that fuzzy structures can be represented within the falling shadow framework. In the other direction, we derived  $T_m$ -type inequalities satisfied by falling subalgebras and falling ideals, yielding  $T_m$ -fuzzy conditions naturally induced by probability bounds on intersections of events. Finally, we discussed  $I$ -fuzzy subalgebras and  $I$ -fuzzy ideals arising from falling inference relations, and we established explicit characterizations in the case  $t = 0.5$  under the diagonal distribution.

These results indicate that falling shadow theory provides a unified and flexible approach for connecting probabilistic random-set models with fuzzy algebraic structures on Hilbert algebras. Several concrete directions merit further study. First, it would be interesting to explore the relationship between falling ideals and congruence relations in Hilbert algebras, in view of the classical correspondence between ideals/filters and congruences (see, e.g., [19]). Second, one may introduce and investigate falling-shadow-based congruences (or random-set representations of congruences) and study how they interact with falling subalgebras and falling ideals under homomorphisms. Third, since fuzzy inference relations depend on the choice of a joint distribution  $P$  and an implication operator  $I$ , it is natural to examine how alternative choices of  $P$  and  $I$  affect the resulting  $I$ -fuzzy structures and to identify distribution- or implication-invariant subclasses. Finally, the present framework can be extended to related implication-based algebras, allowing comparisons of falling shadow representations across different algebraic semantics.

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