A MULTI-DIMENSIONALIZATION OF COMPETITIVE FACILITY LOCATION PROBLEMS

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ABSTRACT. There are many researches for location problems of competitive facilities, e.g., shops and stores, on a line segment or a plane. For applying these studies to other various decision makings, this paper considers competitive facility location problems on multi-dimensional spaces. In order to solve the formulated problem efficiently, first, it is shown that one of its optimal solutions can be found by solving a 0-1 programming problem, next, its efficient solution method is proposed based upon the tabu search algorithm with strategic oscillation. The efficiency of the solution method is shown by applying it to some numerical examples of the multi-dimensional competitive facility location problem.

Keywords: Location, Competitiveness, Multi-dimensional space, Tabu search, Strategic oscillation

1. Introduction. Competitive facility location problem (CFLP) is one of optimal location problems for commercial facilities, e.g., shops and stores, and the objective of a decision maker (DM) for the CFLP is mainly to obtain as many demands for her/his facilities as possible. Mathematical studies on CFLPs are originated by Hotelling [1]. He considered the CFLP under the conditions that (i) customers are uniformly distributed on a line segment, (ii) each of DMs can locate and move her/his own facility at any times, and (iii) all customers only use the nearest facility. As an extension of Hotelling’s CFLP, CFLPs on a plane were studied by Okabe and Suzuki [2], and recently Cabrera et al. [3] applied Hotelling’s CFLP to retail stores location. Wendell and McKelvey [4] assumed that there exist customers on a finite number of points, called demand points (DPs), and they considered the CFLP on a network whose nodes are DPs.

Based upon the CFLP proposed by Wendell and McKelvey, Hakimi [5] considered CFLPs under the conditions that the DM locates her/his facilities on a network where other competitive facilities were already located. Drezner [6] extended Hakimi’s CFLPs to the CFLP on a plane that there are DPs and competitive facilities. For solving Drezner’s CFLP efficiently, Uno et al. [7] proposed an evolutionary multi-agent based search method.

In the above CFLPs, customers choose their facilities by estimating only the distance between them and facilities. Huff [8] defined the attractive function of a facility for
customers by considering not only their distance but also its quality. The CFLP with Huff’s attractive power function is studied by Uno and Katagiri [9], Fernández et al. [10], Bruno and Improta [11], Zhang and Rushton [12] and Uno et al. [13].

The studies of CFLPs are applied to other areas of decision making. Hansen et al. [14] considered voting location problems by applying the studies of CFLPs on a network. Campos and Moreno [15] extended their voting location problems by considering the location of multiple facilities. Yates [16] considered the New York stock exchange by applying Hotelling’s CFLP.

In the former CFLPs and their applications, the space where facilities can be located is at most two-dimensional space. However, there are often three or more decision variables for applying the studies of CFLP to other various areas of decision making. For example, Figure 1 shows that a choice of five electors for three candidates is represented as a CFLP with a three-dimensional space, whose axes are economy, diplomacy, and welfare.

In this paper, we multi-dimensionalize the CFLP by extending Drezner’s CFLP with Huff’s attractive function. Since the formulated problem is a nonlinear programming problem, a strict optimal solution cannot be found directly. We show that one of its optimal solutions can be found by solving a 0-1 programming problem, and propose its solution method to improve the tabu search algorithm with strategic oscillation as well as multidimensional knapsack problems proposed by Hanafi and Freville [17]. For the details of the tabu search algorithms, the readers can refer to the studies of Glover [18] and Reeves [19] and recently Yang and Li [20] proposed a solution method for a railway transportation planning problem based upon the tabu search algorithm. We apply our proposing tabu search algorithm to numerical examples of the CFLPs with random demands, and show its efficiency by comparing it with other solution algorithms.

The construction of this paper is as follows. In Section 2, we formulate the multi-dimensional CFLP. Since it is difficult to solve the formulated problem directly, we show that one of its optimal solutions can be found by solving a 0-1 programming problem in Section 3. In Section 4, we propose an efficient solution method based upon the tabu search algorithm with strategic oscillation by utilizing characteristics of the CFLPs. We show the efficiency of the solution method by applying it to numerical examples of the CFLPs.
CFLPs with random demands in Section 5. Finally, in Section 6, concluding comments and future extensions are summarized.

2. Formulation of Multi-Dimensional CFLP. In the proposed CFLPs, for a positive integer \( \alpha \), we assume that all customers only exist on DPs in \( \mathbb{R}^\alpha \). For convenience sake, by aggregating all customers on the same DP, we regard one DP as one customer.

There are \( n \) DPs in \( \mathbb{R}^\alpha \), and let \( D = \{1, \ldots, n\} \) be the set of indices of the DPs. Let \( m \) be the number of new facilities that the DM locates, and \( k \) be the number of competitive facilities that have been already located in \( \mathbb{R}^\alpha \). The sets of indices of the new facilities and the competitive facilities are denoted by \( F = \{1, \ldots, m\} \) and \( F_C = \{m + 1, \ldots, m + k\} \), respectively.

Let \( u_i \in \mathbb{R}^\alpha \) be the site of DP \( i \), and \( x_j \in \mathbb{R}^\alpha \) and \( q_j > 0 \) be the site and quality of facility \( j \in F \cup F_C \), respectively. Then, the attractive power of facility \( j \) for DP \( i \) is represented as the following function suggested by Huff [8]:

\[
a_i(x_j, q_j) = \begin{cases} 
q_j \frac{1}{||u_i - x_j||^2}, & \text{if } ||u_i - x_j|| > \varepsilon, \\
q_j \frac{1}{\varepsilon^2}, & \text{if } ||u_i - x_j|| \leq \varepsilon,
\end{cases} \tag{1}
\]

where \( \varepsilon > 0 \) is an upper limit of the distance that customers can move without any trouble. It is assumed that all customers only use one facility with the largest attractive power, and if two or more attractive powers appear to be the same one, they use the facility in reverse numerical order of the indices of facilities; that is, in the order of competitive facilities and new facilities.

Let \( x = (x_1, \ldots, x_m) \in \mathbb{R}^{m\alpha} \) be the location of the new facilities. Then we use the following 0-1 variable for representing whether DP \( i \) uses new facility \( j \in F \):

\[
\phi_i^j(x) \triangleq \begin{cases} 
1, & \text{if DP } i \text{ uses the new facility } j, \\
0, & \text{otherwise.}
\end{cases} \tag{2}
\]

Let \( w_i > 0 \) be the buying power (BP) of DP \( i \). New facility \( j \) can obtain the BP \( w_i \) if \( \phi_i^j(x) = 1 \). The objective of the DM is represented as maximizing the sum of BP that all the new facilities obtain. Then, the multi-dimensional CFLP can be formulated as the following programming problem:

\[
\begin{array}{ll}
\text{maximize} & f(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_i^j(x) \\
\text{subject to} & x \in \mathbb{R}^{m\alpha}.
\end{array} \tag{3}
\]

Problem (3) is a nonconvex and nonlinear programming problem. However, even for two-dimensional CFLPs [6, 9, 13], their optimal solutions cannot be found by using general analytic solution methods with differential of the objective function, Kuhn-Tucker conditions, etc. Moreover, Uno and Katagiri [9] showed that an optimal solution of two-dimensional CFLPs could not be found by using heuristic solution methods for nonlinear programming problems, e.g., genetic algorithm for numerical optimization for constrained problem (GENOCOP) [21]. In the next section, we will show that one of the optimal solutions can be found by solving a 0-1 programming problem.

3. Reformulation to 0-1 Programming Problem. First, we describe an outline of proposing solution method for (3), where \( S \) denotes the family of sets of DPs.
Algorithm 1: outline of the solution method for the CFLP.

Step 0: Set $NS \rightarrow S$ and $ES \rightarrow \phi$.

Step 1: Choose a set of DPs $S \in NS$, and set $NS \rightarrow NS\backslash \{S\}$.

Step 2: Find a location of new facilities that can obtain BPs from all DPs in $S$.

Step 3: If a location exists, then $S \rightarrow ES \cup \{S\}$.

Step 4: If $NS \neq \emptyset$, return to Step 1.

Step 5: In $ES$, find the set including DPs whose sum of BPs are maximal. Then a location found in Step 2 for the maximal set is an optimal solution of (3).

Before introducing the solution method, we first omit DPs that each facility $j$ cannot obtain their BPs whenever it locates. For DP $i \in D$, the largest attractive power among all competitive facilities is denoted as follows:

$$a_i^C \triangleq \min_{j \in FC} \{a_i(x_j, q_j)\}. \quad (4)$$

From (1), the set of DPs that new facility $j \in F$ cannot obtain their BPs wherever it is located is represented as follows:

$$D_j^- \triangleq \left\{ i \in D \mid \sqrt{q_j/a_i^C} \leq \varepsilon \right\}. \quad (5)$$

Then, the set of DPs in which there is at least one location of new facility $j$ which can obtain their BPs is denoted by $D_j \triangleq D \backslash D_j^-$. For new facility $j$, let $\tilde{D}_j \subseteq D_j$ be a set of DPs. Let

$$l_{ij} \triangleq \begin{cases} 1, & \text{if } i \in \tilde{D}_j, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Then, $\tilde{D}_j$ can be represented as 0-1 vector $l_j = (l_{1j}, \ldots, l_{nj})$. Note that giving the 0-1 vector $l_j$ for all facilities $j = 1, \ldots, n$ corresponds to Step 1.

Next, we introduce the method for finding a location at Step 2. For new facility $j$ and vector $l_j$ given by the DM, we consider the following problem with an auxiliary variable $r_j \geq 0$:

$$\begin{array}{ll}
\text{maximize} & r_j^2 \\
\text{subject to} & ||u_i - x_j||^2 \leq \frac{q_j}{a_i^C} \cdot r_j, \quad \forall i \in \{\xi | l_{ij} = 1\}, \\
& x \in R^{max}, \quad r_j \geq 0.
\end{array} \quad (7)$$

Let $\left(x_j^{l_j}, r_j^{l_j}\right)$ be an optimal solution of (7). Then, $x_j^{l_j}, j = 1, \ldots, n$ is just the location at Step 2. Note that (7) is a convex programming problem, and then can be solved by using the solution methods for convex programming, such as sequential quadratic programming (SQP) method; for the details of the SQP method, the reader can refer to the book of Nocedal and Wright [22].

At Step 3, the following theorem plays an important role.

**Theorem 3.1.** For new facility $j \in F$, let $\tilde{D}_j$ be a set of DPs given by the DM and $l_j$ be a 0-1 vector corresponding to $\tilde{D}_j$. Then, if $r_j^{l_j} < 1$, the new facility $j$ can obtain all DPs in $\tilde{D}_j$ by locating it at $x_j^{l_j}$.

**Proof:** For the constraint of (7) and $r_j^{l_j} < 1$, it holds that $||u_i - x_j||^2 < q_j/a_i^C$ for all DPs in $\tilde{D}_j$. Then, $a_i^C < q_j/||u_i - x_j||^2$ is satisfied. From (1), this equation means that the attractive power of new facility $j$ is more than that of competitive facilities if the site of new facility $j$ is $x_j^{l_j}$. \qed
By using Theorem 3.1 at Step 3, we can obtain $ES$ used at Step 5. Finally, the following theorem shows that the location given at Step 5 is an optimal location of (3).

**Theorem 3.2.** Let $L = (l_1, \ldots, l_m) \in \{0, 1\}^{mn}$ and $x^L = (x_1^L, \ldots, x_m^L) \in R^{mn}$. Then, there exists $L$ to make $x^L$ be an optimal solution of (3).

**Proof:** Let $x^*$ be an optimal solution of (3). We define the 0-1 matrix $L = (l_1, \ldots, l_m) \in \{0, 1\}^{mn}$, each of whose element for $i \in D$ and $j \in F$ is that $\tilde{l}_{ij} = \phi_i^j (x^*)$. Then, from Theorem 3.1, $x^L$ is also an optimal solution of (3) because $\phi_i^j (x^L) = \phi_i^j (x^*)$ for all $i$, $j$ and $r_{j}^{\tilde{l}} < 1$ for all $j$. This means that $L$ is one of the matrices satisfying the condition of the theorem.

We have already shown that finding an optimal solution of (3) is equivalent to executing Algorithm 1 for (3). Let $r_{\text{max}}^L \triangleq \max \{r_1^L, \ldots, r_m^L\}$. Then, Algorithm 1 can be represented as solving the following 0-1 programming problem to find an optimal matrix $L$ with 0-1 elements:

$$\begin{align*}
\text{maximize} & \quad f(x^L) \\
\text{subject to} & \quad r_{\text{max}}^L < 1, \\
& \quad L \in \{0, 1\}^{mn}.
\end{align*}$$

(8)

In order to find an optimal matrix of (8), we need to solve (7) at $m$ times for all matrices. Because there are $2^{mn}$ matrices in (8), it needs enormous computational times for finding a strict optimal solution of (8). In the next section, we will propose an efficient solution method for (8).

4. **Tabu Search Algorithm with Strategic Oscillation.** First, we introduce an important theorem showing a similarity between the 0-1 programming problem in the previous section and multidimensional knapsack problems. Let $l_j^{k+} \triangleq l_j + e_k$, where $e_k$ is the $k$-th unit vector, and $L_j^{k+} \triangleq t (l_1, \ldots, l_j^{k+}, \ldots, l_m)$.

**Theorem 4.1.** Let $L \in \{0, 1\}^{mn}$ be the matrix that $\tilde{l}_{ij} = \phi_i^j$ for all $i$, $j$ and $x^L$ be an optimal solution of (3). Then, if there exists a $DP$ $k$ satisfying $l_{ij} = 0$ for any new facility $j \in F$, $r_{\text{max}}^{L_j^{k+}} \geq 1$ for any $j$.

**Proof:** We assume that there exists a $DP$ $k$ for a new facility $j$ such that $r_{j}^{k+} < 1$. Then, from Theorem 3.2, new facility $j$ can obtain more values of objective function by being located at $x_j^{k+}$ than $x_j$. This contradicts that $x^L$ is an optimal solution of (3). □

From Theorem 4.1, an optimal solution of (8) exists on the neighborhood of its constraints. This is similar to the multi-dimensional knapsack problems whose optimal solution exists on the neighborhood of its constraints. Moreover, if the matrix $L$ which has many elements that $l_{ij} = 1$, there are many constraints of (7) for obtaining many BPs. Then, if $L$ and $L_j^{k+}$ hold that $r_{\text{max}}^L < 1$ and $r_{\text{max}}^{L_j^{k+}} < 1$, $x^{L_j^{k+}}$ is mostly superior to $x^L$. Similarly, the multidimensional knapsack problems have the character that their objective function values are improved or not changed if an element of solutions is changed from zero to one.

From these two characteristics, we think that solution methods for multidimensional knapsack problems are also efficient for (8). For multidimensional knapsack problems, Hanafi and Freville [17] proposed the tabu search algorithm with strategic oscillation, which we will propose to improve the solution method by utilizing characteristics of CFLP.

The tabu search is one of the local search methods. In our solution method, we define moves from a current solution, denoted by $L^{\text{now}}$, as the increase or decrease of its one
element. The neighborhood of a current solution in (8) is represented as a set of all solutions that can transfer by only one move from the solution. In the tabu search, the next searching solution from $L_{\text{now}}$, denoted by $L_{\text{next}}$, is chosen to the best solution for given criteria, e.g., objective function value, in the neighborhood of a current solution. However, if we use such a search without modification, a circulation of certain chosen moves will occur on the way of search, and then it can only find one local optimal solution. For preventing such a circulation, if a move is chosen in the search, the tabu constraint for its opposite move will be activated for given tenures, called the tabu tenure, and denoted by $T_1$. Then, the activated moves are forbidden to choose in $T_1$ tenures, called tabu, even if the moves make the objective function value of (8) best in all solutions in a neighborhood. Such tabu moves are memorized in the tabu list for the search.

Although the tabu search method has an advantage for searching local areas, the neighborhood of the constraints in the feasible set of (8) with large $m$ and $n$ is too wide to search efficiently. Then, we introduce the strategic oscillation to the tabu search for searching in the neighborhood efficiently. Our proposing solution method is described as follows:

Algorithm 2: Tabu search algorithm with strategic oscillation for CFLP.

**Step 0:** Generate the initial searching solution $L_{\text{now}}$, and initialize the tabu list and other variables. If $r_{L_{\text{max}}} < 1$, then go to Step 4.

**Step 1:** Move $L_{\text{now}}$ to $L_{\text{next}}$ by decreasing an element of $L_{\text{now}}$ with the purpose of decreasing $r_{L_{\text{max}}}$ as much as possible. This step is repeated until it is satisfied.

**Step 2:** Move $L_{\text{now}}$ to $L_{\text{next}}$ with the purpose of improving the objective function value of (8). This step is repeated a given number of times, denoted by $T_2$.

**Step 3:** Move $L_{\text{now}}$ to $L_{\text{next}}$ by decreasing an element of $L_{\text{now}}$ with the purpose of decreasing $r_{L_{\text{max}}}$ as much as possible. This step is repeated until $r_{L_{\text{max}}}$ is less than a certain vector, denoted by $r_{\text{low}}$.

**Step 4:** Move $L_{\text{now}}$ to $L_{\text{next}}$ by increasing an element of $L_{\text{now}}$ with the purpose of improving the objective function value of (8). This step is repeated until it is not satisfied $r_{L_{\text{max}}} < 1$.

**Step 5:** Do the same operations as Step 2.

**Step 6:** Move $L_{\text{now}}$ to $L_{\text{next}}$ by increasing an element of $L_{\text{now}}$ with the purpose of improving the objective function value of (8). This step is repeated until $r_{L_{\text{max}}}$ is more than a certain vector, denoted by $r_{\text{upp}}$.

**Step 7:** If the given terminal conditions are satisfied, then this algorithm is terminated. The obtaining approximate solution is the best solution about the objective function $\|v\|$ of (8) in all searched solutions. Otherwise, return to Step 1.

In the above algorithm, searching all moves for neighborhood of $L_{\text{now}}$ at each step needs many computational times. Then, we streamline the algorithm by utilizing characteristics of the CFLPs for the following two cases.

**Case (i):** For Steps 4 and 6, we propose an estimation function for moves. We consider the following three cases:

- For increasing the objective function value of (8) larger, it is desirable to increase an element of DP with larger BP.
- For increasing the objective function of (7) smaller, it is desirable for facility $j$ to increase its element of DP nearby $x^l_j$.
- It is undesirable to choose the same DP at many times.
5. Numerical Examples. In this section, we will show the efficiency of the solution algorithm in the previous sections by applying it to three examples of the CFLPs. In these examples, the dimension of locating space is $\alpha = 7$. For each example, the numbers of DPs are $n = 40, 50, 60$. The sites of DPs $u_1, \ldots, u_n$ are given in $\alpha$-dimensional Euclid space $[0,10] \times \cdots \times [0,10]$ randomly, and their random BPs $w_1, \ldots, w_n$ are given in $\{1, \ldots, 20\}$ randomly. We give fifteen competitive facilities, that is $k = 15$, and for competitive facility $j \in F_C$, its site $x_j$ and quality $q_j$ are randomly given in $[0,10] \times \cdots \times [0,10]$ and $\{4, \ldots, 9\}$, respectively. In this space, the decision maker locates one facility, that is $m = 1$, whose quality is $q_1 = 8$.

Next, we give parameters about our solution method; for the meanings of parameters of tabu searches, the reader can refer to the book of Reeves [19]. We set the tabu tenure $T_1 = n/2 - 10$. The terminal condition in Step 7 is that the iteration of the tabu search algorithm is false until 10 times. At Step 2, let $T_1 = 10$. At Steps 3 and 6, let $r_{low} = 0.3$ and $r_{upp} = 3$.

For showing the efficiency of our solution method, we compare its computational results with other genetic algorithm; for details of the genetic algorithms, the readers can refer to the studies of Sakawa and Kato [23]. We set generation gap $G = 0.9$, population size $N_{GA} = 150$ and terminal generation $T_{GA} = 2000$. Probabilities of crossover, mutation and inversion are $p_C = 0.9$, $p_M = 0.01$ and $p_I = 0.03$, respectively.

We apply the tabu search algorithm and the genetic algorithm to three examples of the CFLPs, where each of these algorithms is implemented 20 times for each example by using DELL Optiplex GX620 (CPU: Pentium(R) 2.33GHz, RAM: 512MB).

The computational results of solving the CFLPs are shown in Tables 1 and 2, where BEST, MEAN and WORST are the best, mean and worst objective function values of (8) for solutions given by the solution algorithms, respectively. For the case $n = 40$, the tabu search algorithm can obtain the same good solutions to (8) as those of the genetic algorithm with shorter computational times at all 20 times. For the cases $n = 50, 60$, the tabu search algorithm can obtain as well or better solutions for (8) than those of the genetic algorithm with shorter computational times at all 20 times. From all the above cases, our solution method is efficient for the multi-dimensional CFLPs.

6. Conclusions and Future Studies. In this paper, we have proposed a new CFLP on multi-dimensional space. Since the formulated problem is difficult to solve directly, we have shown that the problem can be reformulated as a 0-1 programming problem. Since it needs enormous computational times for finding its strict optimal solution, we have proposed an efficient solution method based upon the tabu search algorithm with strategic oscillation by utilizing characteristics of the CFLPs. The efficiency of the solution method is shown by applying to several examples of the multi-dimensional CFLP.
Table 1. Computational results by the tabu search algorithm with strategic oscillation

<table>
<thead>
<tr>
<th>n</th>
<th>BEST</th>
<th>MEAN</th>
<th>WORST</th>
<th>Mean CPU Times (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>168</td>
<td>168.0</td>
<td>168</td>
<td>107.0</td>
</tr>
<tr>
<td>50</td>
<td>186</td>
<td>186.0</td>
<td>186</td>
<td>154.2</td>
</tr>
<tr>
<td>60</td>
<td>208</td>
<td>208.0</td>
<td>208</td>
<td>358.7</td>
</tr>
</tbody>
</table>

Table 2. Computational results by the genetic algorithm

<table>
<thead>
<tr>
<th>n</th>
<th>BEST</th>
<th>MEAN</th>
<th>WORST</th>
<th>Mean CPU Times (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>168</td>
<td>168.0</td>
<td>168</td>
<td>1103</td>
</tr>
<tr>
<td>50</td>
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<td>60</td>
<td>208</td>
<td>205.7</td>
<td>185</td>
<td>3239</td>
</tr>
</tbody>
</table>

In the proposed multi-dimensional CFLPs, Euclid norm is used for representing the distance from facilities to customers. For applying various types of decision making, there are cases that other norms, e.g., Manhattan norm, supremum norm, etc., are more suitable than Euclid norm. Considering multi-dimensional CFLPs with various types of norms is an important future study.

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