FUZZY PARALLEL SYSTEM RELIABILITY ANALYSIS
BASED ON LEVEL \((\lambda, \rho)\) INTERVAL-VALUED FUZZY NUMBERS

HUEY-MING LEE\(^1\), CHING-FEN FUR\(^2\) AND JIN-SHIEH SU\(^2\)

\(^1\)Department of Information Management
\(^2\)Department of Applied Mathematics
Chinese Culture University
No. 55, Hwa-Kang Road, Yang-Ming-Shan, Taipei 11114, Taiwan
\{hmlee; cff; sjs\}@faculty.pccu.edu.tw

Received March 2011; revised July 2011

ABSTRACT. In this paper, we examine the reliability of a parallel system. We use the
level \((\lambda, \rho)\) interval-valued fuzzy numbers to find the fuzzy reliability of parallel systems
and obtain the estimated reliability of the systems in the fuzzy sense by employing the
signed distance method.

Keywords: Fuzzy reliability, Interval-valued fuzzy number, Statistical data, Confidence
interval, Signed distance

1. Introduction. In this article, we consider the reliability of a parallel system. We
know that in a factory production process if we want to consider the reliability of the
production process experiments are necessary. It is difficult to obtain significant results
with this reliability problem if we only consider a model without using experiments.

Conventional optimization methods assume that all parameters and goals of a model
are precisely known. However, in many practical problems incomplete and unreliable
information exists. Therefore, we use the fuzzy concept to treat this parallel system
reliability problem.

Because the population reliability \(R_j\) of the subsystem \(P_j\) \((j = 1, 2, \ldots, n)\) is unknown,
we can obtain reliable statistical data \(R_{jq}, q = 1, 2, \ldots, n_j\) from the subsystem \(P_j\) in
the parallel system. If we use the average value \(\bar{R}_j\) as the point estimate \(R_j\) from past
statistical data, we will not know the probability of the error \(R_j - \bar{R}_j\). Moreover, the system
reliability may fluctuate around the point estimate \(\bar{R}_j\) during a time interval. It follows
that using the point estimate \(\bar{R}_j\) to estimate the population reliability \(R_j\) is not suitable
for real cases. Therefore, it is more desirable to use the statistical confidence interval.

We use the statistical confidence interval instead of the point estimate. We transfer the
statistical confidence interval into the level \((\lambda, \rho)\) \(i\)-\(v\) fuzzy number. We consider the fuzzy
reliable system through these level \((\lambda, \rho)\) \(i\)-\(v\) fuzzy numbers. We fuzzify the reliability of
parallel systems. Through defuzzifying the fuzzy parallel system reliability using the
signed distance method we obtain a fuzzy reliability estimate in the fuzzy sense.

There are two fundamental hypotheses in conventional reliability theory, namely the
probability assumption and the binary-state assumption. (1\(^\text{st}\)) The probability assumption:
The system behavior is fully characterized in the context of the probability measure.
(2\(^\text{nd}\)) The binary-state assumption: At any given time, the system has only two states. One
is the functioning state and the other is the failed state. In earlier papers [1-4], the authors modified (2\(^\text{nd}\)) to (2\(^*\)) as follows. (2\(^*\)) The fuzzy state assumption: at any given
time, the system has only two states. One is the fuzzy success state and the other is the
fuzzy failure state. In [5], the authors used the \(\alpha\)-cut of level 1 fuzzy numbers to obtain
the interval and find the fuzzy reliability of a serial system and the fuzzy reliability of a parallel system. In [6], they used fuzzy numbers to find the fuzzy reliability of a serial system and the fuzzy reliability of a parallel system. Yao et al. [10] used triangular fuzzy numbers and statistical data to find the fuzzy reliabilities of both systems. Singer [11] did not use the statistical method. He used the L-R type fuzzy number to consider the fuzzy reliability problem.

In this paper, we consider the reliability of a parallel system. We use the statistical confidence interval concept to fuzzify the parallel system using a level \((\lambda, \rho)\) interval-valued fuzzy number and defuzzify the parallel system using the signed distance method. Section 2 presents some properties of fuzzy sets. Section 3.1 uses a statistical point estimate to examine the reliability of the parallel system. In Section 3.2, we use the statistical confidence interval concept converted to level \((\lambda, \rho)\) interval-valued fuzzy number. We give a numerical example in Section 4 and we offer some discussions in Section 5 and conclusions in Section 6.

2. Preliminaries. In order to consider the fuzzy system reliability analysis based on level 1-\(\alpha\) fuzzy numbers and level \((\lambda, \rho)\) interval-valued fuzzy numbers, we provide several definitions as follows [7,9,16]:

**Definition 2.1.** A fuzzy set \(\tilde{A}\) defined on \(R\) is called the level \(\lambda\) triangular fuzzy number if its membership function is
\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x \leq b \\
\frac{c-x}{c-b}, & b \leq x \leq c \\
0, & \text{otherwise},
\end{cases}
\]
where \(a < b < c, 0 < \lambda \leq 1\), then \(\tilde{A}\) is called the level \(\lambda\) fuzzy number and denoted by \(\tilde{A} = (a, b, c; \lambda)\). When \(\lambda = 1\) is called a triangular fuzzy number and denoted by \(\tilde{A} = (a, b, c)\).

**Definition 2.2.** [7]. An interval-valued fuzzy set (i-v fuzzy set for short) \(\tilde{A}\) on \(R\) is given by
\[
\tilde{A} \triangleq \{(x, [\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)])\}, \quad \forall x \in R,
\]
where \(\mu_{\tilde{A}L}(x) \leq \mu_{\tilde{A}U}(x)\) and \(\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x) \in [0, 1]\).

We denote it by \(\tilde{A}(x) = [\mu_{\tilde{A}L}(x), \mu_{\tilde{A}U}(x)], \forall x \in R\), or
\[
\tilde{A} = [\tilde{A}L, \tilde{A}U].
\]

Let \(\tilde{A}L = (a, b, c; \lambda)\) and \(\tilde{A}U = (e, b, h; \rho)\), where \(0 < \lambda \leq \rho \leq 1\) and \(e < a < b < c < h\). Then, the i-v fuzzy set is written as
\[
\tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)]
\]
(1)

We view \(\tilde{A}\) as a level \((\lambda, \rho)\) i-v fuzzy number. The family of all level \((\lambda, \rho)\) fuzzy numbers will then be denoted by
\[
F_{IV}(\lambda, \rho) = \{[(a, b, c; \lambda), (e, b, h; \rho)] | e < a < b < c < h\},
\]
where \(0 < \lambda \leq \rho \leq 1, a, b, c, e, h \in R\).

We let \(\tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)] = [\tilde{A}L, \tilde{A}U] \in F_{IV}(\lambda, \rho)\) and then have the \(\alpha\)-cut of \(\tilde{A}L\) and \(\tilde{A}U\), \(0 \leq \alpha \leq 1\) at the left and right endpoint as follows:
If \( 0 \leq \alpha < \lambda \) then
\[
\tilde{A}_f^L(\alpha) = a + (b - a)\frac{\alpha}{\lambda}, \quad \tilde{A}_f^U(\alpha) = c - (c - b)\frac{\alpha}{\lambda},
\]
and if \( \lambda \leq \alpha \leq \rho \) then
\[
\tilde{A}_f^U(\alpha) = e + (b - e)\frac{\alpha}{\rho}, \quad \tilde{A}_f^L(\alpha) = h - (h - b)\frac{\alpha}{\rho},
\]
Using the decomposition theory with Figure 1, we can represent \( \tilde{A} \) as:
\[
\tilde{A} = \bigcup_{0 \leq \alpha < \lambda} \left( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) ; \alpha \right] \cup \left[ \tilde{A}_r^U(\alpha), \tilde{A}_r^L(\alpha) ; \alpha \right] \right) \cup \bigcup_{\lambda \leq \alpha \leq \rho} \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) ; \alpha \right]
\]
Using the similar method in [9], we discuss the signed distance and ranking of level \( (\lambda, \rho) \) \( i-v \) fuzzy number defined on \( F_{IV}(\lambda, \rho) \). First of all, we consider the definition of the signed distance on \( R \).

**Definition 2.3.** [9]. Let \( a, 0 \in R \). Define \( d_0(a, 0) = a, \ d_0(0, 0) \) is called the signed distance from \( a \) to \( 0 \).

Using Definition 2.3, the signed distances from intervals \( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right] \) and \( \left[ \tilde{A}_r^U(\alpha), \tilde{A}_r^L(\alpha) \right] \) to \( 0 \) can be defined by \( d_0 \left( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right], 0 \right) = \frac{1}{2} \left[ \tilde{A}_r^U(\alpha) + \tilde{A}_r^L(\alpha) \right] \) and \( d_0 \left( \left[ \tilde{A}_r^U(\alpha), \tilde{A}_r^L(\alpha) \right], 0 \right) = \frac{1}{2} \left[ \tilde{A}_f^U(\alpha) + \tilde{A}_f^L(\alpha) \right] \), respectively. Since \( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right] \cap \left[ \tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha) \right] = \emptyset \), from (3) and (4), the signed distance from \( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right] \cup \left[ \tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha) \right] \) to \( 0 \) can be defined by
\[
d_0 \left( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right] \cup \left[ \tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha) \right], 0 \right) = \frac{1}{4} \left[ a + c + e + h + (b - a - c)\frac{\alpha}{\lambda} + (2b - h - e)\frac{\alpha}{\rho} \right].
\]
For each \( \alpha \in [0, \lambda] \), \( \left[ \tilde{A}_f^U(\alpha), \tilde{A}_f^L(\alpha) \right] \leftrightarrow \left[ \tilde{A}_r^U(\alpha), \tilde{A}_r^L(\alpha) ; \alpha \right], \left[ \tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha) \right] \leftrightarrow \left[ \tilde{A}_f^L(\alpha), \tilde{A}_f^U(\alpha) ; \alpha \right] \) and \( 0 \leftrightarrow \hat{0} \) are one to one mappings. Therefore, we define the signed distance
By Definition 2.5, we have

Due to Property 1, we know that

Property 1.

Similarly, when \( \lambda \leq \alpha \leq \rho \), one can define

and

By (6), (7), we give the following definition.

**Definition 2.4.** Let \( \tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)] \in F_{IV}(\lambda, \rho) \). The signed distance from \( \tilde{A} \) to \( \tilde{0} \) is defined as follows:

(1

\[
d(\tilde{A}, \tilde{0}) = \frac{1}{8} \left[ 6b + a + c + 4e + 4h + \frac{3\lambda}{\rho} (2b - h - e) \right]
\]

(2

\[
d(\tilde{A}, \tilde{0}) = \frac{1}{8} [4b + a + c + e + h]
\]

**Definition 2.5.** For \( \tilde{A}, \tilde{B} \in F_{IV}(\lambda, \rho) \), their ordering is defined by

\( \tilde{A} \prec \tilde{B} \iff d(\tilde{A}, \tilde{0}) < d(\tilde{B}, \tilde{0}) \);

\( \tilde{A} \approx \tilde{B} \iff d(\tilde{A}, \tilde{0}) = d(\tilde{B}, \tilde{0}) \).

By the linear order “\(<, =\)” on \( R \) and Definition 2.5, we obtain the following property.

**Property 1.** \( \tilde{A}, \tilde{B}, \tilde{C} \in F_{IV}(\lambda, \rho) \)

(1

One and only one of \( \tilde{A} \prec \tilde{B}, \tilde{A} \approx \tilde{B}, \tilde{B} \prec \tilde{A} \) will occur.

(2

The “\(<, \approx\)” on \( F_{IV}(\lambda, \rho) \) satisfies the following 3 ordering relations.

\[
\begin{align*}
(a) & \quad \tilde{A} \preceq \tilde{A}; \\
(b) & \quad \tilde{A} \preceq \tilde{B}, \tilde{B} \preceq \tilde{A} \Rightarrow \tilde{A} \approx \tilde{B}; \\
(c) & \quad \tilde{A} \preceq \tilde{B}, \tilde{B} \preceq \tilde{C} \Rightarrow \tilde{A} \preceq \tilde{C}.
\end{align*}
\]

Due to Property 1, we know that “\(<, \approx\)” is the linear order on \( F_{IV}(\lambda, \rho) \).

Consider a level \( (\lambda, \rho) \) \( i-v \) fuzzy number \( \tilde{A} \) in (2). Let \( a = b, c = b, \) and \( \lambda = 0 \). (see Figure 1). Therefore, the level \( (\lambda, \rho) \) \( i-v \) fuzzy number \( \tilde{A} \) can be reduced to the level \( \rho \) fuzzy number \( \tilde{A}^{L} \), i.e., the level \( \rho \) fuzzy number is a special case of the level \( (\lambda, \rho) \) \( i-v \) fuzzy number. The family of all level \( \rho \) fuzzy numbers is denoted by

\[
F_{N}(\rho) = \{ (e, b, h; \rho) | e < b < h, e, b, h \in R \}.
\]

By Definition 2.5, we have
Definition 2.6. Let $\tilde{C} = (a,b,c;\rho) \in F_N(\rho)$. The signed distance from $\tilde{C}$ to $\tilde{0}$ is

$$d^*(\tilde{C},\tilde{0}) = \frac{1}{2\rho} \int_0^\rho \left[ \tilde{C}_l(\alpha) + \tilde{C}_r(\alpha) \right] d\alpha = \frac{1}{4}(2b+a+c).$$

Same as Definition 2.5, we also have

Definition 2.7. For $\tilde{B}, \tilde{C} \in F_N(\rho)$, their ordering is defined by

$\tilde{B} < \tilde{C} \iff d^*(\tilde{B},\tilde{0}) < d^*(\tilde{C},\tilde{0});$

$\tilde{B} \approx \tilde{C} \iff d^*(\tilde{B},\tilde{0}) = d^*(\tilde{C},\tilde{0}).$

Definition 2.8. Let $\tilde{A} = [\tilde{A}^L, \tilde{A}^U], \tilde{B} = [\tilde{B}^L, \tilde{B}^U] \in F_{IV}(\lambda, \rho)$, and let $k \in R$. We define the following operations:

$$\tilde{A} \oplus \tilde{B} = [\tilde{A}^L \oplus \tilde{B}^L, \tilde{A}^U \oplus \tilde{B}^U],$$

$$k\tilde{A} = [k\tilde{A}^L, k\tilde{A}^U],$$

$$\tilde{A}^L = (a_{L1}, a_{L2}, a_{L3}; \lambda), \quad \tilde{A}^U = (a_{U1}, a_{U2}, a_{U3}; \rho),$$

$$\tilde{B}^L = (b_{L1}, b_{L2}, b_{L3}; \lambda), \quad \tilde{B}^U = (b_{U1}, b_{U2}, b_{U3}; \rho),$$

$$\tilde{A}^L \oplus \tilde{B}^L = (a_{L1} + b_{L1}, a_{L2} + b_{L2}, a_{L3} + b_{L3}; \lambda),$$

$$\tilde{A}^U \oplus \tilde{B}^U = (a_{U1} + b_{U1}, a_{U2} + b_{U2}, a_{U3} + b_{U3}; \rho),$$

$$k\tilde{A}^L = \begin{cases} (ka_{L1}, ka_{L2}, ka_{L3}; \lambda), & \text{if } k > 0 \\ (ka_{L3}, ka_{L2}, ka_{L1}; \lambda), & \text{if } k < 0 \end{cases}.$$
Let \( R_j \) (unknown), \( j = 1, 2, \ldots, n \) be the population reliability of the subsystem \( P_j \). The reliability of the parallel system is then
\[
1 - \prod_{j=1}^{n} (1 - R_j).
\]  \( \text{(11)} \)

3.2. **Reliability based on level \((\lambda, \rho)\) \(i\)-\(v\) fuzzy numbers.** When we measure the reliability \( R_j \) of the subsystem \( P_j \) \( (j = 1, 2, \ldots, n) \) for a parallel system a general error will occur. It would be more realistic to say that the reliability will be around \( R_j \). It must be notified that saying “around \( R_j \)” is to use the fuzzy language. It appears that using a fuzzy set to express its meaning is better. Due to the unknown probability of error in point estimation \( \tilde{R}_j \) and \( R_j \) is unknown, we use the confidence interval of \( \tilde{R}_j \) instead.

![Figure 3. Level \((\lambda, \rho)\) interval-valued fuzzy number \( \tilde{R}_j \)](image-url)

The \((1 - \alpha) \times 100\%\) confidence interval of \( R_j \) is
\[
[\tilde{R}_j - \Delta_{1j}, \tilde{R}_j + \Delta_{1j}], \quad j = 1, 2, \ldots, n,
\]  \( \text{(12)} \)
where \( \Delta_{1j} = t_{n_j-1}(\alpha_1) \frac{s_j}{\sqrt{n_j}}, \Delta_{4j} = t_{n_j-1}(\alpha_2) \frac{s_j}{\sqrt{n_j}}, 0 < \alpha_j < 1, j = 1, 2, \alpha_1 + \alpha_2 = \alpha, \) \( 0 < \alpha < 1, 0 < \Delta_{1j}, \Delta_{4j} < \tilde{R}_j \) and \( s_j^2 = \frac{1}{n_j-1} \sum_{q=1}^{n_j} (R_{jq} - \tilde{R}_j)^2, j = 1, 2, \ldots, n. \)

Let \( T \) be the random variable of \( t \) distribution with \( n_j - 1 \) degrees of freedom. \( t_{n_j-1}(\alpha_k) \) then satisfies \( p(T \geq t_{n_j-1}(\alpha_k)) = \alpha_k, k = 1, 2. \) Since the \((1 - \alpha) \times 100\%\) confidence interval in \( \text{(12)} \) is an interval, the decision makers choose a point within the interval to estimate \( R_j \). We find that if the chosen point is \( \tilde{R}_j \), then there is no deference between the chosen point and the point to estimate \( \tilde{R}_j \) and the error is definitely 0. We obtain the maximum confidence level. Let it be \( \rho = 1 - \alpha. \) According to the same reason as above, we have level \( \rho \) fuzzy number \( \text{(13)} \) corresponding to \( \text{(12)} \) as follows:
\[
\tilde{R}_j^U = (\tilde{R}_j - \Delta_{1j}, \tilde{R}_j, \tilde{R}_j + \Delta_{4j}; \rho), \quad j = 1, 2, \ldots, n,
\]  \( \text{(13)} \)
where \( 0 < \tilde{R}_j - \Delta_{1j} < 1, \quad \alpha_1 + \alpha_2 = \alpha, \quad \rho = 1 - \alpha, \quad j = 1, 2, \ldots, n. \) \( \text{(14)} \)

Similarly, let \( 0 < \alpha < \beta < 1, 0 < \alpha_j < \beta_j < 1, j = 1, 2 \) and \( \alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta. \) We have \((1 - \beta) \times 100\%\) confidence interval
\[
[\tilde{R}_j - \Delta_{2j}, \tilde{R}_j + \Delta_{3j}], \quad j = 1, 2, \ldots, n,
\]  \( \text{(15)} \)
where \( \Delta_{2j} = t_{n_j-1}(\beta_1) \frac{s_j}{\sqrt{n_j}} \) and \( \Delta_{3j} = t_{n_j-1}(\beta_2) \frac{s_j}{\sqrt{n_j}}. \)

In corresponding to confidence interval above, we have the level \( \lambda \) triangular fuzzy number as follows:
\[
\tilde{R}_j^L = (\tilde{R}_j - \Delta_{2j}, \tilde{R}_j, \tilde{R}_j + \Delta_{3j}; \lambda), \quad j = 1, 2, \ldots, n,
\]  \( \text{(16)} \)
where

\[ 0 < \tilde{R}_j - \Delta_{2j} < 1, \quad \beta_1 + \beta_2 = \beta, \quad \lambda = 1 - \beta, \quad j = 1, 2, \ldots, n. \]

Since \( 0 < \lambda < \rho \), from (13) and (16), we have the following level \((\lambda, \rho)\) \(i-v\) fuzzy numbers

\[ \tilde{R}_j = [\tilde{R}_j^L, \tilde{R}_j^U] \in F_{IV}(\lambda, \rho), \quad j = 1, 2, \ldots, n. \] (17)

**Theorem 3.1.** Using the level \((\lambda, \rho)\) \(i-v\) fuzzy numbers \(\tilde{R}_j = [\tilde{R}_j^L, \tilde{R}_j^U], j = 1, 2, \ldots, n\) in (17), we obtain the fuzzy reliability of the parallel system as follows:

\[
\begin{aligned}
&\hat{\tilde{R}}_1 \otimes \hat{\tilde{R}}_2 \otimes \cdots \otimes \hat{\tilde{R}}_n = \left[ \prod_{j=1}^{n} (\tilde{R}_j - \Delta_{2j}) \right] \otimes \left[ \prod_{j=1}^{n} (\tilde{R}_j^L) \right] \otimes \left[ \prod_{j=1}^{n} \left( \tilde{R}_j + \Delta_{3j} \right) ; \lambda \right] \times \\
&\left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j + \Delta_{1j}) \right) \otimes \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j) \right) \otimes \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j - \Delta_{4j}) ; \rho \right) 
\end{aligned}
\] (18)

**Note 1.** The fuzzy reliability of the serial system is

\[
\begin{aligned}
\hat{\tilde{R}}_1 \otimes \hat{\tilde{R}}_2 \otimes \cdots \otimes \hat{\tilde{R}}_n &= \left[ \prod_{j=1}^{n} (\tilde{R}_j - \Delta_{2j}) \right] \otimes \left[ \prod_{j=1}^{n} \tilde{R}_j \right] \otimes \left[ \prod_{j=1}^{n} (\tilde{R}_j + \Delta_{3j}) ; \lambda \right] \times \\
&\left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j + \Delta_{1j}) \right) \otimes \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j) \right) \otimes \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j - \Delta_{4j}) ; \rho \right) 
\end{aligned}
\]

**Theorem 3.2.** Using Definition 2.4, we can defuzzify (18) to get the estimate reliability of the parallel system in the fuzzy sense as follows:

\[
\frac{1}{2} \left( \hat{\tilde{R}}_1 \otimes \left[ \hat{\tilde{R}}_2 \otimes \cdots \otimes \hat{\tilde{R}}_n \right] \right), 0 \right) = \frac{1}{16} \left[ 6b_2 + a_2 + c_2 + 4p_2 + 4r_2 + \frac{3\lambda}{\rho} (2b_2 - p_2 - r_2) \right] 
\] (19)

where

\[
\begin{aligned}
a_2 &= 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j + \Delta_{2j}), \quad b_2 = 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j), \quad c_2 = 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j - \Delta_{3j}), \\
p_2 &= 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j + \Delta_{1j}), \quad r_2 = 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j - \Delta_{4j}).
\end{aligned}
\]

4. **Numerical Example.**

**Example 4.1.** In [8,11], they considered the following problem. Two grinding machines are working next to each other. What is the possibility that people coming into the vicinity of the machines will be injured mainly by getting a chip into their eyes? The most endangered persons are the operators who are obliged to wear safety glasses but often fail to do this. Further endangered are persons coming into the vicinity of the machines, who are persons who bring and carry away items, and others entering the area for other reasons.

The basic events contributing to the accident are as shown in Table 1.

\[
\begin{aligned}
U &= F + G + H, \quad V = C + D, \quad Z = E \times U \times V, \quad X = A + B + Z. \\
\text{Let } \alpha &= 0.02, \quad \alpha_1 = 0.011, \quad \alpha_2 = 0.009, \quad \beta = 0.2, \quad \beta_1 = 0.12, \quad \beta_2 = 0.08, \quad t_9(\alpha_1) = 2.7017, \\
t_9(\alpha_2) &= 2.9068, \quad t_9(\beta_1) = 1.2698, \quad t_9(\beta_2) = 1.5630.
\end{aligned}
\]
Table 1. The basic events contributing to the accident [8,11]

<table>
<thead>
<tr>
<th>j</th>
<th>Symbol</th>
<th>Populations reliability</th>
<th>Sample size</th>
<th>Sample mean</th>
<th>Sample standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>( R_A )</td>
<td>( n_A = 10 )</td>
<td>( \bar{R}_A = 0.1 )</td>
<td>( S_A = 0.004 )</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>( R_B )</td>
<td>( n_B = 10 )</td>
<td>( \bar{R}_B = 0.2 )</td>
<td>( S_B = 0.004 )</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>( R_C )</td>
<td>( n_C = 10 )</td>
<td>( \bar{R}_C = 0.8 )</td>
<td>( S_C = 0.010 )</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>( R_D )</td>
<td>( n_D = 10 )</td>
<td>( \bar{R}_D = 0.6 )</td>
<td>( S_D = 0.010 )</td>
</tr>
<tr>
<td>5</td>
<td>E</td>
<td>( R_E )</td>
<td>( n_E = 10 )</td>
<td>( \bar{R}_E = 0.9 )</td>
<td>( S_E = 0.020 )</td>
</tr>
<tr>
<td>6</td>
<td>F</td>
<td>( R_F )</td>
<td>( n_F = 10 )</td>
<td>( \bar{R}_F = 0.5 )</td>
<td>( S_F = 0.004 )</td>
</tr>
<tr>
<td>7</td>
<td>G</td>
<td>( R_G )</td>
<td>( n_G = 10 )</td>
<td>( \bar{R}_G = 0.6 )</td>
<td>( S_G = 0.004 )</td>
</tr>
<tr>
<td>8</td>
<td>H</td>
<td>( R_H )</td>
<td>( n_H = 10 )</td>
<td>( \bar{R}_H = 0.7 )</td>
<td>( S_H = 0.001 )</td>
</tr>
</tbody>
</table>

Table 2. Two endpoints in Figure 4

<table>
<thead>
<tr>
<th>j</th>
<th>Symbol</th>
<th>( \bar{R}<em>j - \Delta</em>{1j} )</th>
<th>( \bar{R}<em>j + \Delta</em>{1j} )</th>
<th>( \bar{R}<em>j - \Delta</em>{2j} )</th>
<th>( \bar{R}<em>j + \Delta</em>{3j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>0.0966</td>
<td>0.1037</td>
<td>0.0984</td>
<td>0.1020</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>0.1966</td>
<td>0.2037</td>
<td>0.1984</td>
<td>0.2020</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>0.7915</td>
<td>0.8092</td>
<td>0.7960</td>
<td>0.8049</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>0.5915</td>
<td>0.6092</td>
<td>0.5960</td>
<td>0.6049</td>
</tr>
<tr>
<td>5</td>
<td>E</td>
<td>0.8829</td>
<td>0.9184</td>
<td>0.8920</td>
<td>0.9099</td>
</tr>
<tr>
<td>6</td>
<td>F</td>
<td>0.4966</td>
<td>0.5037</td>
<td>0.4984</td>
<td>0.5062</td>
</tr>
<tr>
<td>7</td>
<td>G</td>
<td>0.5966</td>
<td>0.6037</td>
<td>0.5984</td>
<td>0.6062</td>
</tr>
<tr>
<td>8</td>
<td>H</td>
<td>0.6991</td>
<td>0.7069</td>
<td>0.6996</td>
<td>0.7005</td>
</tr>
</tbody>
</table>

From Table 2, we get the fuzzy reliability of C and D:

\[
\bar{R}_C = [(0.7960, 0.8, 0.8049; 0.8), (0.7915, 0.8, 0.8092; 0.98)];
\]

\[
\bar{R}_D = [(0.5960, 0.6, 0.6049; 0.8), (0.5915, 0.6, 0.6092; 0.98)].
\]

According to Theorem 3.1, we obtain the fuzzy reliability of the parallel system \( V = C + D \), as follows:
fuzzy sense to consider the fuzzy reliability problem. The L-R type fuzzy number operations are as

\[
\left( \hat{1} \ominus \tilde{R}_C \right) \otimes \left( \hat{1} \ominus \tilde{R}_D \right)
\]

\[
= [(0.1951, 0.2, 0.2040; 0.8), (0.1908, 0.2, 0.2085; 0.98)] \otimes [(0.3951, 0.4, 0.4040; 0.8), (0.3908, 0.4, 0.4085; 0.98)]
\]

\[
= [(0.0771, 0.08, 0.0824; 0.8), (0.0746, 0.08, 0.0852; 0.98)]; \quad \tilde{R}_V = [(0.9176, 0.92, 0.9229; 0.8), (0.9148, 0.92, 0.9254; 0.98)].
\]

Similarly,

\[
\tilde{R}_U = \hat{1} \ominus \left[ \left( \hat{1} \ominus \tilde{R}_F \right) \otimes \left( \hat{1} \ominus \tilde{R}_G \right) \otimes \left( \hat{1} \ominus \tilde{R}_H \right) \right]
\]

\[
= [(0.9395, 0.94, 0.9418; 0.8), (0.9389, 0.94, 0.9412; 0.98)].
\]

Due to Note 1, we get the fuzzy reliability of the serial system \(Z = E \times U \times V\) as follows:

\[
\tilde{R}_E = [(0.8920, 0.9, 0.9099; 0.8), (0.8829, 0.9, 0.9184; 0.98)];
\]

\[
\tilde{R}_Z = \tilde{R}_E \otimes \tilde{R}_U \otimes \tilde{R}_V
\]

\[
= [(0.7690, 0.7783, 0.7909; 0.8), (0.7583, 0.7783, 0.7999; 0.98)].
\]

According to Theorem 3.1, we get the fuzzy reliability of the parallel system \(X = A + B + Z\), as follows:

\[
\tilde{R}_A = [(0.0984, 0.1, 0.1020; 0.8), (0.0966, 0.1, 0.1037; 0.98)];
\]

\[
\tilde{R}_B = [(0.1984, 0.2, 0.202; 0.8), (0.1966, 0.2, 0.2037; 0.98)];
\]

\[
\tilde{R}_X = \hat{1} \ominus \left[ \left( \hat{1} \ominus \tilde{R}_A \right) \otimes \left( \hat{1} \ominus \tilde{R}_B \right) \otimes \left( \hat{1} \ominus \tilde{R}_Z \right) \right]
\]

\[
= [(0.1434, 0.1596, 0.1669; 0.8), (0.1428, 0.1596, 0.1754; 0.98)].
\]

By Definition 2.4, we defuzzify \(\tilde{R}_X\) and get the estimate reliability of the system in the fuzzy sense

\[
\frac{1}{2} d \left( \tilde{R}_X, \hat{0} \right) = 0.1588.
\]

5. Discussions.

(A) The crisp case is a special case of fuzzy case.

If \(\alpha_k \to 0.5, \beta_k \to 0.5, k = 1, 2\), then \(\alpha \to 1, \beta \to 1, t_{n-1}(\alpha_k) \to 0, t_{n-1}(\beta_k) \to 0, k = 1, 2\). The result in Theorem 3.1 will then be reduced to

\[
\hat{1} \ominus \left[ \left( \hat{1} \ominus \tilde{R}_1 \right) \otimes \left( \hat{1} \ominus \tilde{R}_2 \right) \otimes \cdots \otimes \left( \hat{1} \ominus \tilde{R}_n \right) \right]
\]

\[
\to \left[ \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j) \left( 1 - \tilde{R}_j \right), 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j) \left( 1 - \tilde{R}_j \right); \lambda \right), \right.
\]

\[
\left. \left( 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j), 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j), 1 - \prod_{j=1}^{n} (1 - \tilde{R}_j); \rho \right) \right].
\]

(B) The comparison of this article and [11].

In [11], the author did not use the statistical method. He used L-R type fuzzy number to consider the fuzzy reliability problem. The L-R type fuzzy number operations are as
follows:

\[
(m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR};
\]

\[
(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} = (m - n, \alpha + \delta, \beta + \gamma)_{LR};
\]

\[
(m, \alpha, \beta)_{LR} \bullet (n, \gamma, \delta)_{LR} = (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}.
\]

The fuzzy reliability of the subsystem \( P_j \) is \( \hat{R}_j = (m_j, \alpha_j, \beta_j)_{LR} \).

The equation above only denotes \( n - 1 \) approximations. He did not defuzzify the system and did not find the estimated reliability of the parallel system in the fuzzy sense.

(C) The comparison of this article and [5].

The \( \alpha \)-level set of triangular fuzzy number \( \hat{A} = (a_1, a_2, a_3) \) is

\[
\hat{A}_\alpha = [a_\alpha^0, a_\alpha^3] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3], \quad \alpha \in [0, 1].
\]

In [5], the fuzzy reliability of the parallel system is

\[
\hat{P} = 1 - \prod_{i=1}^{n} (1 - \hat{P}_i) = [1, 1] - \prod_{i=1}^{n} ([1, 1] - [a_{i1}^\alpha, a_{i3}^\alpha])
\]

\[
= \left[ 1 - \prod_{i=1}^{n} (- (a_{i2} - a_{i1})\alpha + 1 - a_{i1}), 1 - \prod_{i=1}^{n} ((a_{i3} - a_{i2})\alpha + 1 - a_{i3}) \right].
\]

To consider the \( i \)th fuzzy reliability, we let \( \hat{P}_i = (a_{i1}, a_{i2}, a_{i3}) \). Using the decomposition theorem, we have

\[
\hat{P}_i = \bigcup_{0 \leq \alpha \leq 1} \left[ (a_{i2} - a_{i1})\alpha + a_{i1}, -(a_{i3} - a_{i2})\alpha + a_{i3}; \alpha \right], \quad i = 1, 2, \ldots, n
\]

and

\[
\hat{P}_1 \otimes \hat{P}_2 \otimes \cdots \otimes \hat{P}_n = \bigcup_{0 \leq \alpha \leq 1} \left[ \prod_{i=1}^{n} ((a_{i2} - a_{i1})\alpha + a_{i1}), \prod_{i=1}^{n} ((a_{i3} - a_{i2})\alpha + a_{i3}); \alpha \right].
\]

Therefore, in [5], the author used only one \( \alpha \) for the decomposition to treat the problem.

6. Conclusions.

(A) If we want to consider the reliability of a factory production process, experimental work is required. We can get the reliability statistical data \( R_{jq}, q = 1, 2, \ldots, n_j \) of the subsystem \( P_j \) in the parallel system (see Figure 2). It is difficult to obtain significant results with this reliability problem if we consider only a model that does not include experiments.

For the subsystem \( P_j \), from the statistical data \( R_{jq}, q = 1, 2, \ldots, n_j \) we can find their average value. Let \( \tilde{R}_j = \frac{1}{n_j} \sum_{q=1}^{n_j} R_{jq} \in [0, 1] \). This is a point estimate of \( R_j \) in the statistical sense. We use it as the point estimate of the reliability of \( P_j \). Because the probability of the error between the point estimate \( \tilde{R}_j \) and \( R_j \) is unknown, we use the confidence intervals Equation (12) and Equation (15) in Section 3.2 instead.

Since the interval is not a value, it is not convenient in the calculation. Therefore, we convert the confidence interval into a level \((\lambda, \rho)\) \( i-v \) fuzzy number Equation (17) in Section 3.2. We use the signed distance to defuzzify and then have the estimate of the reliability of \( P_j \) in the fuzzy sense.

(B) This paper is better than articles in [5,6] in the real application.

(B-1) In [6], for the reliability of the subsystem \( P_j \), the author used the triangular fuzzy number \( \tilde{R}_j = (m_j - \alpha_j, m_j, m_j + \beta_j) \). He did not discuss how to determine \( m_j, \alpha_j \) and \( \beta_j \). Therefore, we cannot apply it to a real problem. In our paper, the level \((\lambda, \rho)\) \( i-v \) fuzzy
numbers $\tilde{R}_j = \left[ \tilde{R}^L_j, \tilde{R}^U_j \right]$ in Equation (17) are derived from statistical data and confidence intervals, where $\tilde{R}_j$ is the average value of statistical data $\tilde{R}_{jq}$, $q = 1, 2, \ldots, n_j$, $S^2_j$ is the sample variance. Both of them are from statistical data with $\alpha$, $\alpha_1$ and $\alpha_2$ provided by the decision maker. The values $m_j$, $\alpha_j$ and $\beta_j$ in [6] correspond to $\tilde{R}_j$, $t_{n_j-1}(\alpha_1) \frac{S_j}{\sqrt{n_j}}$ and $t_{n_j-1}(\alpha_2) \frac{S_j}{\sqrt{n_j}}$ in this paper. Therefore, this paper is better than [6] in reality applications.

(B-2) In [5], they used the $\alpha$-level set $[(a_{j2} - a_{j1})\alpha + a_{j1}, -(a_{j3} - a_{j2})\alpha + a_{j3}]$ of the triangular fuzzy number $(a_{j1}, a_{j2}, a_{j3})$ to consider the reliability of subsystem $P_j$, where $(a_{j2} - a_{j1})\alpha + a_{j1}$ is the left hand side of $\alpha$-cut and $-(a_{j3} - a_{j2})\alpha + a_{j3}$ is the right hand side of $\alpha$-cut, $0 < \alpha \leq 1$, $j = 1, 2, \ldots, n$, as shown before. They did not discuss how to determine $a_{j1}$, $a_{j2}$ and $a_{j3}$ as stated in (B-1). Therefore, it cannot be applied to a real problem. Similarly, as shown in (B-1), $a_{j1}$, $a_{j2}$ and $a_{j3}$ correspond to $\tilde{R}_j$, $t_{n_j-1}(\alpha_1) \frac{S_j}{\sqrt{n_j}}$ and $t_{n_j-1}(\alpha_2) \frac{S_j}{\sqrt{n_j}}$ in this paper. Therefore, this paper is better than [5] in reality applications.

Acknowledgment. The authors would like to express their sincere gratitude toward the anonymous referees for their great comments.

REFERENCES