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A BULLWHIP TYPE OF INSTABILITY INDUCED BY TIME VARYING TARGET INVENTORY IN PRODUCTION CHAINS

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Abstract. *We present an analytic investigation of the bullwhip effect developing in production-distribution chains. All common considered causes of the effect are excluded and the only mean to induce a bullwhip type of instability is the adoption of an inventory replenishment policy involving a variable target level. The policy is designed to maintain a safety stock that is proportional to the actual demand. In order to achieve our goals we develop a particular discrete model of supply chains by introducing some fresh concepts into the field of supply chain stability. The basic idea is to derive an update scheme describing the status of the whole chain over the entire time-space (period-stage) domain of interest. We prove that the strategy of demand driven target inventory inherently leads to an instability developing in the chain, which is precisely a manifestation of the bullwhip effect. Following the identification of the source and the nature of the instability, we propose a new production plan, which is stable and does not exhibit the bullwhip effect at all. Thus, the amplitude of the variation of the production rate never exceeds the amplitude of the oscillation of the market demand within the entire supply chain.*
Keywords: Order-up-to policy, Supply chain, Stability, Bullwhip effect

1. **Introduction.** In the increasing global competition supply chain management has become a critical factor [1]. In principle, a supply chain is composed of a set of sequentially connected production or distribution units, giving rise to information and material flow between its members. At the beginning of the chain the market demand acts as the driver of the dynamic processes evolving in the system, while at the end the producer of raw materials provides the material input. Typically, the market demand for a product exhibits different types of variations:

- small stochastic oscillations around a mean value (random changes of the demand),
- rapid changes with large amplitude (caused by promotions),
- slow variations with large amplitude (due to seasonal, economical tendencies).

A common observation is that these variations have an amplifying effect on the order and the production rate upwards in the chain. A large number of reports and case studies indicate that even small perturbations of the market demand can cause huge variations in the order and the production rates. Practical experience implies that the higher a stage is situated in the supply chain, the larger the variations are. This is the essence of the bullwhip effect, which is an undesired feature of modern supply chains. As a result, the members of the chain might experience the following consequences:

- inefficient utilization of production and inventory resources,
- increased shipping costs,
- decreasing level of customer service,

- lost revenues.

It may even happen that a factory has to close temporarily, just to be forced to increase its regular production by several factors in the following period. Due to these reasons the investigation of the bullwhip effect has been in the focus of extensive research for more than a decade. Probability theory [2, 4], control theory [5, 6], delay differential equations [7], linear stability analysis [8], high order autoregressive demand process [9], fuzzy [10, 11] and chaos perspectives [12] are put into work to handle the phenomenon. Although it seems that in a number of models the underlying driving forces have been identified, the complete understanding of their nature, and the treatment of the bullwhip effect still remain the subject of active research. The work of Lee et al. [2] elegantly clarified that the bullwhip effect can be induced by a number of conceptually different triggering mechanisms. The possible triggers identified so far are:

- demand signal forecasting,
- non-zero lead time,
- order batching,
- rationing game,
- promotions.

What rarely receives attention is that these five causes do not automatically trigger the bullwhip effect. Instead, the mechanisms are brought to play by *human decisions*, which are channelled through inventory management and ordering policies. These policies contain a number of parameters, which govern the significance of the individual triggers, or even may introduce new triggering mechanisms. Therefore, it is also reasonable to consider the inventory replenishment policy itself as a possible cause of the bullwhip effect. A general replenishment policy widely employed in the literature is due to John et al. [3]. It is called the automatic pipeline, inventory and order-based production control system (APIOBPCS). According to APIOBPCS, the placed order is the sum of the following three components:

- an average demand, usually smoothed over a given time period (its role is to cover the current demand),
- a fraction of the difference between the actual and the target inventory level (its role is to move the actual inventory level towards its target value),
- a fraction of the difference between the actual and the target work in process (WIP) (its role is to move the WIP towards its target value).

It is a quite popular choice to assume that the target level of the inventory is constant, calculated by some statistical methods. In the present paper we omit this assumption to improve the realism of the model. Our interest goes exclusively into the triggering of the bullwhip effect by a variable target inventory depending on the actual demand. Since in the considered model this inventory management policy can be directly translated into a production plan, we will use the latter term throughout this study. All other causes listed above, including delay effects (consequently WIP terms) are neglected. Nevertheless, as results indicate, the bullwhip effect clearly appears even in this fairly simple scenario.

The particular model of supply chains that we study in this paper was investigated experimentally by Buchmeister and his co-workers in [13, 14, 15]. Our choice has the following three reasons. First, in the model the only possible triggering mechanism of the bullwhip effect is the production plan involving a variable target stock. Second, the authors of [13, 14, 15] based their work on a series of spread sheet simulations, which has an experimental nature with limited concluding power. We improve upon their results by means of mathematical analysis. Third, the model lends itself for a simple introduction of a new methodology that we propose for the study of supply chain stability. In the presented model the market demand is served by the first stage of K sequentially linked stages of a supply chain (see Figure 1). Buchmeister et al. focused on the appearance of the bullwhip effect in 2-stage and 4-stage chains, as the market demand was following different patterns (sudden jump, periodic, etc.). They measured the magnitude of the bullwhip effect for different safety stock keeping policies. The work of [13] demonstrated that on the selected test cases one particular stock keeping policy resulted in a considerably reduced bullwhip effect. These studies gave clear evidence that the particular choice of the production plan (stock keeping policy) in itself is capable of inducing the bullwhip effect, and that it may play a crucial role in the partial suppression of the instability. Additionally, these results imply the following questions that were not addressed by the authors.

- What is the very source of the bullwhip effect in the given model?
- Why is one stock keeping policy better than another?
- Is there an optimal stock keeping policy in some sense?
- Is it possible to eliminate the bullwhip effect completely?
- If so, what is the price to pay?

In order to answer these questions, we need to pursue the construction of an appropriate mathematical framework that might lead us to a deeper understanding of the dynamics of the particular supply chain model under consideration.

The aim of the present paper is threefold. First, we build a discrete analytic model of a supply chain involving production units at its stages. The chain is operated under the conditions of a periodic review method where lead time does not appear directly. The only trigger capable to induce the bullwhip effect is the particular choice of the adopted production plan that involves a variable safety stock depending on the actual demand. Second, we put the model into work, identify the origin of the bullwhip effect, provide a method for its complete elimination, and investigate the related price to pay. Third, we present a fresh concept in the field of supply chain methodology that is different both from the nowadays popular control theoretic approach [5, 6] and from the framework of linear delay differential equations [7]. The basic idea is to formulate a discrete update scheme that determines the status of the whole supply chain over a discrete time-space (period-stage) domain and study its stability properties. By exploiting this idea, a simple, compact, yet efficient discrete model can be obtained, which exhibits several observed characteristics of supply chains. In particular it provides precise conditions for controlling the emergence of the bullwhip effect.

The outline of this paper is the following. In Section 2, we develop a discrete analytic model of supply chains. In Section 3, we point out the reason why the bullwhip effect is an inherent feature even of this very simple scenario, and we propose a new production plan to eliminate the instability and discuss the related price. Numerical experiments supporting the theoretical results are presented in Section 4. Finally, in Section 5 some implications of this new strategy are discussed and practical conclusions are drawn.

2. **The Supply Chain Model.** In this section we present a discrete model of supply chains by taking a new perspective, and provide its mathematical description. The chain is composed of the market, and K sequentially linked stages ($K \in N^+$). Since the role of the stages may differ in specific models, we do not give them individual names, such as manufacturer and retailer. Instead, according to the labelling conventions of Figure 1, the individual stages are identified by index k (market: $k = 0$, fully operating stages: $K = 1, 2, \dots, K$). Buchmeister et al. [13, 14, 15] studied the $K = 2$ and the $K = 4$ cases experimentally by means of spread sheet simulation. In these papers the market demand

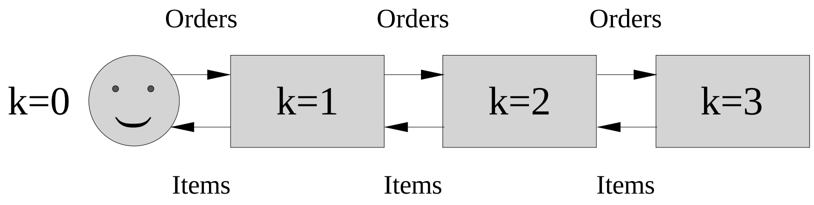

FIGURE 1. A market served by a supply chain with $K = 3$ stages

($k = 0$) is observed and satisfied by a manufacturer ($k = 1$), which is served by one or three sequentially linked suppliers in the $K = 2$ and the $K = 4$ cases, respectively.

The dynamics of this supply chain model is investigated in a periodic review fashion over a temporal interval subdivided into $N + 1$ non-overlapping periods. These periods are labelled by $n \in [0, 1, \dots, N]$. During period n stage k experiences a demand of d_n^k items, it keeps a starting stock of $\bar{s}_n^k$ and a finishing stock of s_n^k items, and maintains a production of p_n^k items. In the given model we assume, that the production procedure of an item manufactured at stage k requires one part (another type of item) that is produced at stage $k + 1$. Thus, the production of p_n^k number of items at stage k requires the same number of parts ordered from and produced by stage $k + 1$. The events taking place during period n at stage k are the followings. The production unit observes its demand, calculates the targeted safety stock at the end of the period, orders the necessary number of parts from stage $k + 1$, receives the parts, performs the production, fills the demand. Clearly, it is a simplistic model in the sense, that lead time and production time is not taken into direct consideration. However, the conclusions provide important insight into the dynamics of this basic model that can be extended to more complex cases.

Stage 0, the market itself has a special role providing the initial demand driving the rest of the chain. In fact, it has neither its own stock nor its demand. For later use we consider that stage 0 has a p_n^0 production only, that is equal to the market demand experienced directly by stage 1 ($p_n^0 = d_n^1$). As a convention, subscript n exclusively labels time periods, while superscript k stands for the k-th stage of the chain. The temporal evolution, i.e., the dynamics of the supply chain is governed by the following strategy, referred to as production plan 1.

Production plan 1 (`PP1`): *During period n for all $k > 0$ the objective of the production is to satisfy the experienced demand d_n^k, and to ensure, that at the end of the period the safety stock equals to αd_n^k, if possible, where $\alpha \geq 0$ is a factor providing the size of the desired safety stock in the unit of the experienced demand.*

This production plan was tested in [13, 14, 15] for different values of α, with the following assumption referring to the initial status of the chain:

Assumption 2.1. *Initially, during period $n = 0$, the chain is in equilibrium, i.e.,*

$$p_0^k = d_0^1 \quad \textit{and} \quad \bar{s}_0^k = s_0^k = \alpha d_0^1. \tag{1}$$

Assumption 2.1 implies, that as long as the demand stays constant, the status of the whole chain remains stationary. Thus, the safety stock and the production rate preserve their respective initial value for all $k > 0$. We keep this assumption until the end of the present section, where we investigate its relevance.

The reasoning behind `PP1` is as follows. If the market demand increases, stage 1 has to increase its production rate first to fill the needs, and second, to accommodate a sufficient safety stock matching the higher demands. In the cited references the authors considered the $\alpha = 1$, $\alpha = 2$ and other cases, where the value of α depends on k. They also employed more complex production plans prone to the bullwhip effect that we do not investigate here.

PP1 has an important implication. If the demand is sufficiently small compared with the starting stock, then there is no production needed in period n at stage k at all to fill the needs. This is a highly undesired phenomenon, since it leads to a large amount of unexploited production potential during a whole period, i.e., temporarily closing the production unit at the given stage. More precisely, if

$$d_n^k \le \frac{\bar{s}_n^k}{1+\alpha} \text{ then } p_n^k = 0 \text{ and } s_n^k = \bar{s}_n^k - d_n^k, \tag{2}$$

should be required. In this case the finishing stock will not be the same as desired by PP1. Otherwise, $s_n^k = \alpha d_n^k$ can be satisfied with a production rate obtained from the following balance equation:

$$p_n^k + \bar{s}_n^k = d_n^k + s_n^k. \tag{3}$$

Observe that the above equation contains four quantities, d_n^k, p_n^k, $\bar{s}_n^k$ and s_n^k, which are not all independent. The dependencies are described below.

The demand experienced at the k-th stage is given by the production rate at the $k-1$-th stage for $k > 0$:

$$d_n^k = p_n^{k-1}. \tag{4}$$

The starting stock during period n at stage k equals to the finishing stock of the same stage at the preceding period:

$$\bar{s}_n^k = s_{n-1}^k. \tag{5}$$

As long as there is no vanishing production present in the system, according to PP1 the finishing stock must be equal to α in the unit given by the demand:

$$s_n^k = \alpha d_n^k. \tag{6}$$

Equations (4)-(6) can be used to eliminate all variables from Equation (3) but the production rate itself, leading to the following two equivalent forms of the production equation:

$$p_n^{k+1} = p_n^k + \alpha\left(p_n^k - p_{n-1}^k\right), \tag{7}$$

$$p_n^{k+1} = p_n^k(1+\alpha) - \alpha p_{n-1}^k. \tag{8}$$

These equations describe the dynamics of the supply chain governed by PP1, as long as relation (2) is not valid. Thus, they provide a solid ground for the analytic investigation to understand the properties of this model.

Equations (7) and (8) define a so-called *update scheme* for the production rate within the entire supply chain during all the periods of interest, that we refer to as the discrete *time-stage* or *period-stage* domain $\Omega = [0, 1, \ldots, N] \times [0, 1, \ldots, K]$. From mathematical point of view, we deal with an initial value-boundary value problem, describing the advection of information (the value of the production rate) along the supply chain. The objective is to determine the unknown values of p_n^k over domain Ω from the known values of p_0^k and p_n^0 (see Figure 2).

The market demand p_n^0 provides the initial solution for update scheme (7), while the production rate p_0^k at period 0 yields the corresponding boundary conditions. Based on these values, we can compute the production at stage 1 for all the time periods of interest, starting at $n = 1$ and eventually ending up at $n = N$. With these values at hand, we can march forward along the stages, and compute the production at the first, second, and

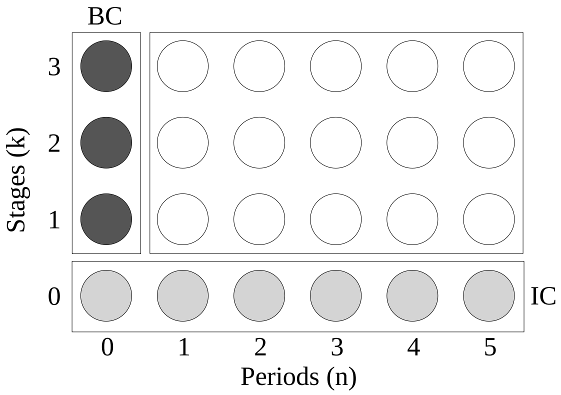

FIGURE 2. The time-stage domain Ω of a supply chain with $K = 3$ stages. Light circles: known initial conditions (market demand). Dark circles: known boundary conditions (status of the supply chain at $n = 0$). Empty circles: unknown states to be computed. IC: initial conditions. BC: boundary conditions.

eventually at the K-th stage. Thus, we obtain the production within the entire domain Ω.

2.1. **Notes on the equivalence of `PP1` and update scheme (7).** Whenever Assumption 2.1 is satisfied at $n = 0$, `PP1` and update scheme (7) lead to identical evolution of the supply chain, provided, that the production does not have to be stopped due to excessive safety stock or very low demand. On the other hand, if the production has to be stopped, i.e., when relation (2) holds, update scheme (7) yields negative number for the production rate, which is an invalid value. From this point the update scheme fails to follow the operation of `PP1`. The explanation is quite straightforward. During the operation of `PP1` condition (2) is explicitly checked and the corresponding adjustments are made, if needed. On the other hand, Equation (6) was used during the construction of update scheme (7), while the checking of condition (2) is omitted. Nevertheless, this fact does not diminish the potential value of the update scheme, since even in this form it gives valid answers regarding the stability of the supply chain subject to `PP1`.

This situation becomes slightly more involved, when Assumption 2.1 is not satisfied. In this case `PP1` and update scheme (7) lead to different evolution of the supply chain. However, once again the stability property of the chain subject to `PP1` is well modelled by (7). It does not bring much practical benefit to study all the different theoretical possibilities, apart from one particular scenario that we describe in the following.

Consider Assumption 2.1. It may happen, that the starting and the finishing stock deviates from their equilibrium value by the same constant offset δ, i.e.,

$$p_0^k = d_0^1 \ \text{ and } \ \bar{s}_0^k = s_0^k = \alpha d_0^1 + \delta. \tag{9}$$

In this case `PP1` will eliminate this difference, because it is directly detected by the plan. However, the enforcement of condition (6) is weakly implemented into update scheme (7), therefore it is *not sensitive* to such symmetric offsets in the safety stocks. The scheme will evolve the production rate *exactly* the same way as there would be no offset at all ($\delta = 0$). This feature can be easily understood by examining the structure of the balance Equation (3) serving the basis for the derivation of update scheme (7). Rearranging Equation (3) leads to

$$p_n^k = d_n^k + (s_n^k - \bar{s}_n^k). \tag{10}$$

The last equation reveals, that the evolution of the production rate depends on the *difference* of the finishing and the starting stocks, thus any symmetric offset will be simply ignored by the update scheme.

In conclusion, the evolution of the production rate will be exactly identical to the $\delta = 0$ case. On the other hand, the safety stock evolves the same way as for $\delta = 0$, apart from the constant offset permanently superposed onto the starting and the finishing stocks.

Remark 2.1. *The above argument justifies, that the presence of symmetric offsets in the safety stocks does not remove the chain from the state of equilibrium if it is evolved by the update scheme. However,* `PP1` *removes the chain from the state of equilibrium in this case.*

3. Stability Analysis of the Supply Chain – The Origin of the Bullwhip Effect.

Upon a closer examination of Equation (7) we realize that formally it is equivalent to the first-order accurate upwind discretization of the one-dimensional scalar advection equation. A von Naumann type of stability analysis [16] of such schemes reveals, that update scheme (7) is unstable for $\alpha > 0$, i.e., the initial solution p_n^0 is amplified without bounds as k increases. This numerical instability is precisely a manifestation of the bullwhip effect. Consequently, the bullwhip effect is an unavoidable, natural consequence of `PP1` with the choice of $\alpha > 0$. The stability analysis also yields, that update scheme (7) is stable if and only if

$$-1 \leq \alpha \leq 0, \tag{11}$$

thus, the bullwhip effect can be avoided by stabilizing the system with the appropriate choice of α.

3.1. The meaning of negative value of $\boldsymbol{\alpha}$.

We have just seen, that our update scheme derived from `PP1` can only be stable for non-positive values of α. However, the $\alpha < 0$ case is meaningless in the context of `PP1`, since the size of the safety stock cannot have a negative value. Additionally, Assumption 2.1 simply cannot be satisfied at $n = 0$ because of the very same reason. Obviously, the starting and the finishing stocks during period 0 have to have a non-negative value, say S_0:

$$\bar{s}_0^k = s_0^k = S_0. \tag{12}$$

Compared with the value of $s_0^k = \alpha d_0^1 < 0$ required by the satisfaction of Assumption 2.1, defining the state of initial equilibrium, the values of $\bar{s}_0^k$ and s_0^k contain the following offset

$$\delta = S_0 + |\alpha| d_0^1, \tag{13}$$

where the absolute sign is introduced to emphasis that $\delta > S_0$. According to the argument at the end of Section 2.1, in this case our update scheme evolves the production rate as Assumption 2.1 would be satisfied, while the safety stock will experience the permanent offset δ. In conclusion, for an arbitrary initial stock our update scheme will result in a stable evolution of both the production rate p_n^k and the safety stock that is equal to

$$s_n^k = \alpha p_n^{k-1} + \delta, \tag{14}$$

where the offset is given by Equation (13).

3.2. **The boundedness of the production rate.** Considering Equation (8) it is immediate to observe, that our update scheme is not only stable, but is monotone as well under constraint (11) in the sense of Spekreijse [17]. Indeed, the unknown production rate at stage $k+1$ is a convex sum of the known values of the production rate at stage k:

$$p_n^{k+1} = \sum_{m=1}^{N} \gamma_m^k p_m^k, \tag{15}$$

where

$$\gamma_m^k \geq 0 \text{ and } \sum_{m=1}^{N} \gamma_m^k = 1, \quad \forall k \in [1, 2, \ldots, K]. \tag{16}$$

As a consequence,

$$\max_{n,k} |p_n^k - p_0^k| \leq \max_n |p_n^0 - p_0^0|, \tag{17}$$

meaning, that the largest deviation of the production rate within the entire supply chain does never exceed the largest deviation of the market demand from their respective initial state. Thus, the production rate stays bounded independently on the size of the supply chain and on the length of the considered temporal interval! In other terms, the bullwhip effect has been completely eliminated at its root to obtain a stable supply chain.

3.3. **A new, stable production plan.** In the light of our results it is worth to formulate a new production plan, which is free of the bullwhip effect. This plan guarantees the stability of the entire supply chain regardless the number of stages and temporal intervals considered.

Production plan 2 (`PP2`)**:** *During period n for all $0 < k \leq K$ the production rate at stage k is $(1+\alpha)d_n^k - \alpha d_{n-1}^k$, where $-1 \leq \alpha \leq 0$.*

Remark 3.1. *Note, that update scheme (8) was derived from* `PP1`*, while* `PP2` *actually* is *update scheme (8) with the choice of (11).*

Remark 3.2. *Observe that* `PP1` *is based on an order-up-to philosophy for the safety stock, where the target level follows the actual demand. On the contrary,* `PP2` *is based on the requirement of stability, order-up-to level is not considered.*

Remark 3.3. `PP2` *implies, that the supply chain operates in a stable manner as long as the production rate is kept between the last and the present observed demand at all its stages, i.e.,*

$$\min(d_{n-1}^k, d_n^k) \leq p_n^k \leq \max(d_{n-1}^k, d_n^k). \tag{18}$$

3.4. **The price of stability.** Although the choice of `PP2` yields a stable, bullwhip-free supply chain, where orders, productions and safety stocks stay bounded, it has its limitations. Inserting offset (13) into Equation (14) we obtain that if relation

$$d_n^k - d_0^1 > \frac{S_0}{|\alpha|} \tag{19}$$

holds, then the finishing stock at stage k during period n becomes negative. This feature, that would never happen for the unstable choice of $\alpha = 1$ in `PP1`, can be foreseen from Remark 3.2. Indeed, `PP2` does not enforce the replenishment of the safety stock, not even when it becomes small. If the demand experienced at stage k deviates from the initial market demand d_0^1 with a larger extent than the threshold given by relation (19), the safety stock becomes negative. In the considered model this feature translates to unfilled demand during the given period. Clearly, by increasing the level of the initial safety stock the scheme can handle larger variations of the demand.

4. **Numerical Results.** In order to demonstrate the performance of the new production plan PP2 proposed in the previous section, we perform two variants of the test case presented in [14] for $K = 5$. In both cases, during period 0 the chain is situated in an equilibrium. The market demands 100 items and all the stages of the supply chain keep a safety stock of 100 items and maintain a production of 100 items per period. This stationary state is perturbed by two different demand patterns investigated in the following subsections.

4.1. **Step profile.** First, the stationary state is perturbed by a sudden drop of the market demand to 95 items per period. With the choice of $\alpha = 1$ for PP1 and $\alpha = -1$ for PP2 we computed the production at all the stages for $N = 6$ periods. The largest deviation of the production rate and the safety stock from their respective initial values are presented in Table 1. The numbers indicate that the bullwhip effect is clearly induced by PP1. It is worth to have a look at the production rates at the fifth stage, presented in Table 2. At every odd period the production unit at stage 5 has to be closed, while at the second period it has to produce 3.4 times the actual demand of the market. In contrast, the production rate and the safety stock stays stable by our new production plan (PP2). Even at the fifth stage it exhibits well balanced behavior as indicated by the data presented in Table 2.

TABLE 1. Sudden drop of demand (step profile). The largest deviation of the production rate and the safety stock from their respective initial values for PP1 and PP2.

	PP1	PP2
$\max_{n,k} \lvert p_n^k - p_0^k \rvert$	240%	5%
$\max_{n,k} \lvert s_n^k - s_0^k \rvert$	140%	5%

TABLE 2. Sudden drop of demand (step profile). The production rate at stage 5 for PP1 and PP2.

Periods	1	2	3	4	5	6
PP1	0	340	0	140	0	95
PP2	100	100	100	100	100	95

4.2. **Random variation.** Next, we investigate a realistic demand pattern, involving a random variation with uniform distribution over interval [95, 105]. The stationary state is perturbed by this random variation of the demand with an amplitude of 5%. We keep on using the value of $\alpha = 1$ for PP1 and $\alpha = -1$ for PP2. The production rates and the actual levels of the safety stocks are plotted on Figures 3 and 4, respectively, for two randomly chosen subsequent periods n and $n+1$. The largest deviation of the production rate and the safety stock from their respective initial values are presented in Table 3. The production rates at the fifth stage are presented in Table 4. The results indicate the presence of an even more pronounced bullwhip effect induced by PP1. It is noteworthy, that the production rate reaches 4.7 times the market demand, while the level of the safety stock is more than 5 times the value of the actual demand of the market. Just like in the previous case, the production rate and the safety stock stays bounded by our new production plan PP2.

TABLE 3. Random variation of demand with uniform distribution. The largest deviation of the production rate and the safety stock from their respective initial values for PP1 and PP2.

	PP1	PP2
$\max_{n,k} \lvert p_n^k - p_0^k \rvert$	368%	5%
$\max_{n,k} \lvert s_n^k - s_0^k \rvert$	413%	5%

TABLE 4. Random variation of demand with uniform distribution. The production rates at two subsequent periods are presented at stage 5 for PP1 and PP2.

Periods	n	$n+1$
PP1	468	0
PP2	99	98

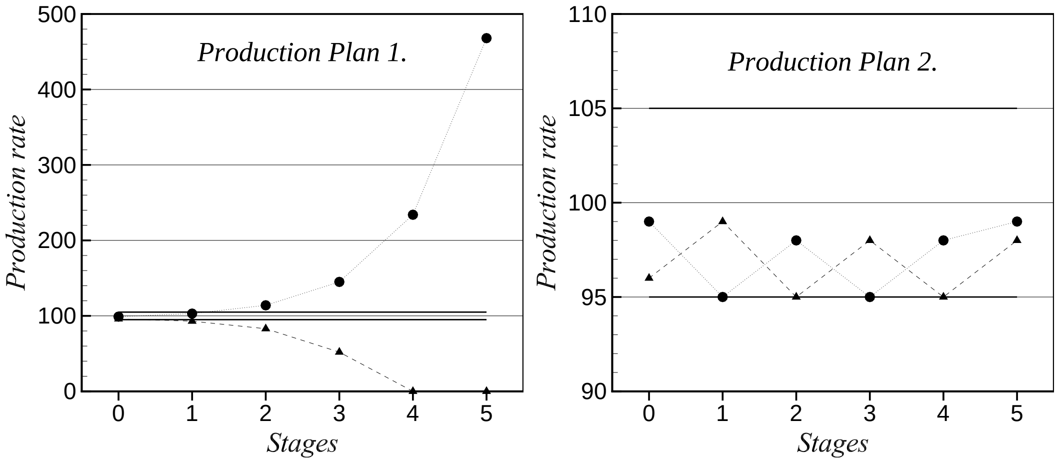

FIGURE 3. Production rates inside a $K = 5$ stage supply chain for a random market demand with uniform distribution over interval [95, 105]. Left: results obtained by PP1. Right: results obtained by PP2. Thick solid lines represent the range of demand variation. Dotted line with circles: values at period n. Dashed line with triangles: values at period $n + 1$.

5. **Discussion and Conclusions.** In this paper we investigated a particular model of supply chains that is free of all the commonly accepted mechanisms responsible for the induction of the bullwhip effect. Amongst others we excluded the Forrester effects [18], i.e., the mechanisms of demand signal forecasting and non-zero lead time. For the sake of simpler analysis most theoretical studies employ an order-up-to policy with a constant target inventory level for the safety stock. Here we adopted a more realistic choice designed for maintaining a safety stock proportional to the actually observed demand. The anticipated benefit of this philosophy labelled by PP1 is clear: it intends to decrease the cost of inventory holding by following the changes of the demand.

Buchmeister et al. [13, 14, 15] investigated this model by means of spread sheet simulations. In order to understand the properties of the dynamic processes reported in these experimental studies, we developed a mathematical model of supply chains operating in discrete time. The goal is to study supply chain dynamics in general, and to investigate

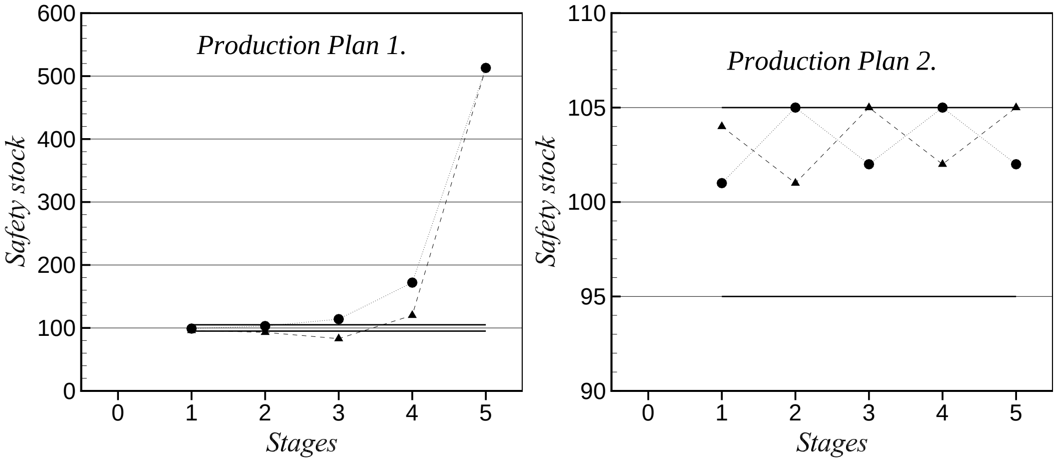

FIGURE 4. Safety stock at the members of a $K = 5$ stage supply chain for a random market demand with uniform distribution over interval [95, 105]. Left: results obtained by PP1. Right: results obtained by PP2. Thick solid lines represent the range of demand variation. Dotted line with circles: values at period n. Dashed line with triangles: values at period $n + 1$.

the stability property of the presented policy in particular. The method is conceptually different from the popular control theoretic approach. The main difference is that we build a discrete update scheme that drives the status of the whole supply chain within the entire period-stage domain of interest, instead of using z-transform on the production equations.

Our results implies, that the strategy of maintaining a variable target inventory level depending on the actual demand properly serves the market. However, the policy is inherently prone to the bullwhip effect, thus unavoidably induces erratic behaviour within the entire supply chain. We presented a test case demonstrating, that at one period the production rate is several times larger than the market demand itself, while in the subsequent period the production has to be stopped completely. This undesired feature leads to highly inefficient use of the production potential at one period, and expensive forced capacities in another.

Upon detecting the very source of this bullwhip type of instability, we derived an alternative production plan (PP2) focusing on the design concept of stability, rather than on the principle of demand following target inventory. Our plan guarantees bounded production rates and safety stock levels along the whole chain, where the magnitude of the variations are bounded by the magnitude of the market demand's oscillation. In order to provide some insight into the underlying mechanism, we rewrite the RHS of update scheme (8) in terms of the demand experienced at stage k as

$$p_n^k = (1 + \alpha)\, d_n^k - \alpha d_{n-1}^k, \text{ where } -1 \le \alpha \le 0. \tag{20}$$

This form of the update scheme clarifies that in PP2 the production rate is a convex linear combination of the present and the last experienced demand. Thus, any production rates satisfying relation (18) results in stable operation. Otherwise, the chain is determined to be unstable. Considering the value of α, there are three cases of interest, the two limiting cases and the $-1 < \alpha < 0$ case.

The $\boldsymbol{\alpha = 0}$ case. If $\alpha = 0$, then $p_n^k = d_n^k$. Thus, stage k is not concerned with past information. Its production strictly follows the demand experienced in the current period. In other terms, all stages produce as much items as their current demand requires. This case corresponds to the so-called on the fly production, which, in principle does not require a safety stock. However, the stages may still keep a stock, which stays at a constant level. This scenario might appeal to highly responsive and flexible production units, where the maintenance of a safety stock is not desired for some reason.

The $\boldsymbol{\alpha} = \mathbf{1}$ case. At the other end of the stability zone $\alpha = -1$, so the production becomes $p_n^k = d_{n-1}^k$. Stage k is not concerned with current information. Instead, it adjusts its production capacity to match exactly the demand experienced in the previous period. In this case all stages have to keep an inventory that will fluctuate in a stable manner. However, if the amplitude of the demand fluctuation becomes too large as indicated by relation (19), PP2 leads to negative safety stock that is translated to unfilled demands.

The $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$ case. If $0 < \alpha < 1$, then the production at stage k is obtained as a linear interpolation between d_{n-1}^k and d_n^k.

The following question naturally emerges: what should be the initial level of the safety stock for PP2 in a particular case? Consider a scenario when the stochastic variation of the market demand is bounded by Δ with probability

$$P\left(\max_n |d_n^1 - d_0^0| \leq \Delta\right). \tag{21}$$

Relation (19) directly implies that the safety stock stays non-negative (the market demand can be fully served) with the very same probability of (21) if the initial safety stock is chosen to be

$$S_0 = \Delta|\alpha|. \tag{22}$$

Nevertheless, if the demand is prone to slow variations with large amplitude on the long run, the safety stock managed by PP2 will eventually need to be periodically updated.

In conclusion, PP1 has the ability to modify the target inventory by following the changes of the market demand, however, it will unavoidably destabilize the whole chain. On the other hand, while PP2 guarantees stable operation, it is prone to the inability to serve the market if the demand is subject to variations with large amplitude.

In our future work we intend to investigate even deeper implications of this analytic approach, and to present some case studies taken from actual market scenarios. Due to apparent similarities between the two systems, recent results in the field of traffic flow control [8, 19, 20] seem also worth to consider for the investigation of supply chain dynamics.

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