

## ADAPTIVE FILTERING FOR STOCHASTIC RISK PREMIA IN BOND MARKET

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**ABSTRACT.** *We consider the adaptive filtering problem for estimating the randomly changing risk premium and its system parameters for zero-coupon bond models. The term structure model for a zero-coupon bond is formulated including the stochastic risk-premium factor. We specify our observation data from the yield curve and bond data which are used to hedge some option claims. For the fixed system parameters, the Kalman filter for the risk-premium and the factor process is constructed first. Secondly, by using the parallel filtering technique and resampling technique commonly used in particle filters, the on-line estimation algorithm for model parameters is constructed. Some simulation studies are finally presented.*

**Keywords:** Adaptive parameter estimation, Kalman filter, Bond market, Term structure model, Stochastic risk premium

**1. Introduction.** In finance, “hedging” is one of the most important mechanisms of reducing the investment risk, and is an essential part of modern financial activities. In order to carry out hedging, we first specify the model structure of the underlying assets. We then need to identify the model parameters of the considered model from the observed data. This leads to inconsistency, as the model is formulated in the risk-neutral world for pricing purposes, while model parameters are estimated in the real world. This difficulty can be resolved by introducing market price of risk terms. In this paper, where we consider the bond market, we develop an adaptive method for parameter identification and estimation of the market price of risk for subsequent hedging procedure.

The arbitrage-free approach to modeling the term structure of interest rates is initiated and clearly developed by Heath, Jarrow and Morton [15], known as the HJM framework. This model is based on the specification of term structure of forward rates in terms of the initial forward rate curve and the forward rate volatility. For calibrating this volatility, there exist many approaches, e.g., in [7, 9, 11, 14, 16]. Recently, starting from a simple short rate model, we proposed a general affine term structure model for bonds with infinite noise sources in [1, 3, 5]. This modeling enabled us to identify model parameters through the Kalman filter without any need to add artificial noises. From the practical study of bond returns, Cochrane and Piazzesi [12] reported that there exists “predictability” in bond returns. This phenomenon may be explained by introducing some stochastic risk premium term on bonds and Collin-Dufresne and Goldstein propose a new dynamics of this risk premium term with feedback of noise sources from the forward rate dynamics

in [13]. In this paper, we include this stochastic premium dynamics in the factor model treated in [3]. Noting that the stochastic risk-premium is not a tradable asset, we should estimate this process from the market data for hedging some option claims. Although the yield curve data from the market is used in [3], we also include some bond data used for hedging some option claims. The main purpose of this paper is to establish the estimation procedure for the stochastic risk-premium from the augmented data of yield curve and some bond data and jointly estimating the systems parameters included in the risk-premium dynamics.

The market studied here becomes incomplete due to the fact that the market price of risk is a stochastic process. This implies that we cannot perfectly hedge the risk in the usual sense [15]. One possibility is to introduce the mean-variance hedging procedure [17]. Although in this paper we do not treat the mean-variance hedging procedure, the on-line estimation procedure developed in this paper can be directly incorporated in the mean-variance hedging.

In Section 2, we review the term structure model proposed in [3] with the stochastic risk-premium given in [13]. The observation model for yield curve data is presented in Section 3 and the statistical identification method for the volatility of the forward rate is presented in [4]. Section 4 is devoted to augmenting the yield curve, and two bond data, that are used to construct a portfolio, as the new observation data. In Section 5, we construct the adaptive Kalman filter to estimate the stochastic risk premium, factor process and unknown systems parameters by using the parallel filter algorithm given in [8, 10]. In Section 6, we present some simulation results for demonstrating the feasibility of the proposed estimation procedure. In the final section, we conclude the paper.

**2. Forward Rate Model with Stochastic Risk Premia.** Let  $(\Omega, \mathcal{F}, \mathcal{P})$  be a probability space endowed with the filtration  $\mathcal{F}_{t \geq 0}$ . The time variable  $t$  is defined on  $[0, t_f]$  and the time-to-maturity variable  $x$  is defined on the extended region  $\tilde{G} = ]0, \hat{T} + t_f[$ . We work with the usual Sobolev space  $H^1(\tilde{G})$  and the inner product  $(\cdot, \cdot)$  with its norm  $\|\cdot\|$  in  $L^2(\tilde{G})$ . Now we present the instantaneous forward rate  $f(t, x)$  as

$$df(t, x) = \frac{\partial f(t, x)}{\partial x} dt + \left( \frac{1}{2} \frac{d}{dx} \tilde{q}(x) - \lambda(t) q_\lambda(x) \right) dt + dw(t, x) \quad (1)$$

$$f(0, x) = f_o(x) \quad (2)$$

where  $w(t, x)$  denotes the two parameter Brownian motion with

$$\begin{aligned} \tilde{E}\{w(t, x_1)w(t, x_2)\} &= q(x_1, x_2)t, \\ \tilde{q}(x) &= \int_0^x \int_0^x q(x_1, x_2) dx_1 dx_2, \end{aligned} \quad (3)$$

and  $\lambda(t)$  is the stochastic market price of risk multiplied by some deterministic function  $q_\lambda(x)$ .

We list some typical cases for the risk-premium term:

1.  $\lambda \equiv 0$ . In this case, the measure  $\mathcal{P}$  becomes a risk neutral measure and there exists no arbitrage opportunity.
2.  $\lambda q_\lambda(x) = C_\lambda \frac{d}{dx} \tilde{q}(x)$  for some constant  $C_\lambda$ . In this case, we can apply the MLE method to identifying this constant  $C_\lambda$  from [3].
3.  $\lambda(t)$  is a solution of the stochastic differential equation with some noise sources which are independent of the forward rate noise. The estimation procedure has been proposed in [4].

4.  $\lambda(t)$  is a solution of a stochastic differential equation with feedback of noise sources from the forward rate model. This modeling comes from the evidence of “predictability” in bond returns studied in [12].

In this paper, we study the situation stated in case 4 above and choose the model specified by Collin-Dfresne and Goldstein in [13]:

$$d\lambda(t) = (a\lambda(t) + b)dt + (\sigma_\lambda, dw(t, \cdot)), \quad \lambda(0) = \lambda_o \tag{4}$$

where  $a$  and  $b$  are constants and  $\|\sigma_\lambda\|^2 < \infty$ .

**Theorem 2.1.** *Under*

(C-1)  $f_o \in L^2(\Omega, H^1(G))$

(C-2)  $\int_G \frac{\partial^2 q(x,y)}{\partial x \partial y} \Big|_{y=x} dx < \infty$

(C-3)  $\lambda_o \in L^2(\Omega; R^1)$

and

(C-4)  $q_\lambda \in H^1(G), \sigma_\lambda \in L^2(G), a$  and  $b$  are constants, we have

$$f \in L^2 \left( \Omega; C([0, t_f]; H^1([0, \hat{T}])) \right), \tag{5}$$

$$\lambda \in L^2(\Omega; C([0, t_f]; R^1)). \tag{6}$$

**Proof:** From (C-1) and  $\int_G q(x, x)dx < \infty$ , we can show that (4) has a unique solution in (6). By using the technique used in [1, 3], (5) can be derived.

**3. Yield Curve Data and Identification of Volatility.** We set continuously compounded yields on zero-coupon bonds with fixed time-to-maturity as our new observation data:

$$y_i(t) = \frac{1}{\tau_i} \int_0^{\tau_i} f(t, x)dx, \text{ for } \tau_1 < \tau_2 < \dots < \tau_m. \tag{7}$$

Define

$$Y(t) = [y_i(t)]_{m \times 1}.$$

Then

$$dY(t) = H_\delta f(t, \cdot)dt - \lambda(t) \left[ \frac{1}{\tau_i} \int_0^{\tau_i} q_\lambda dx \right]_{m \times 1} dt + \frac{1}{2} \left[ \frac{1}{\tau_i} \tilde{q}(\tau_i) \right]_{m \times 1} dt + \left[ \frac{1}{\tau_i} \int_0^{\tau_i} dw(t, x)dx \right]_{m \times 1}, \tag{8}$$

where

$$H_\delta[\cdot] = \left[ \frac{1}{\tau_i} \int_G (\delta(x - \tau_i) - \delta(x))(\cdot)dx \right]_{m \times 1}.$$

By using Ito’s formula, we have

$$\frac{Y(t)Y'(t) - \int_0^t dY'(s)Y(s) - \int_0^t Y(s)dY'(s)}{t} = \left[ \frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} q(x_1, x_2)dx_1 dx_2 \right]_{m \times m}. \tag{9}$$

Noting that the volatility kernel  $\frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} q(x_1, x_2)dx_1 dx_2$  can be obtained from (9), and setting some functional form of  $q$  with some unknown parameters, we can identify this kernel by using the least square method as already established in [4].

**4. Filtering and Augmented Observation.** In addition to the yield curve data, we also observe the bond data which are used for hedging the option claims. Here we consider the European call option  $(P(T_m, T_M) - K)^+$  where  $K$  denotes the strike price and  $P(T_m, T_M)$  denotes the bond price at time  $T_m$  with the maturity  $T_M$ . We construct the following portfolio:

$$V(t, \theta) = P(t, T_m)x_0 + \int_0^t \theta(s)dP(s, T_M), \tag{10}$$

where  $P(t, T_m)$  ( $P(t, T_M)$ ) denotes the bond price at present time  $t$  with the maturity  $T_m$  ( $T_M$ ) and is given by

$$P(t, T_m) = \exp \left\{ - \int_0^{T_m-t} f(t, x)dx \right\}. \tag{11}$$

Furthermore,  $x_0$  is an initial investment for the bond  $P(t, T_m)$  and the portfolio  $\theta(t)$  denotes the amount of the bond  $P(t, T_M)$  which is kept at time  $t$ . Now we observe the whole processes  $P(t, T_m)$  and  $P(t, T_M)$  for  $0 \leq t \leq T_m$ . Hence, we construct the following data:

$$\tilde{Y}(t) = -\log \frac{P(t, T_M)}{P(t, T_m)}.$$

Noting that

$$\tilde{Y}(t) = \int_{T_m-t}^{T_M-t} f(t, x)dx,$$

we have

$$\begin{aligned} d\tilde{Y}(t) &= \frac{1}{2} \left( \int_{T_m-t}^{T_M-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2)dx_1dx_2 + 2 \int_0^{T_m-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2)dx_1dx_2 \right) dt \\ &\quad + \int_{T_m-t}^{T_M-t} dw(t, x)dx - \lambda(t) \int_{T_m-t}^{T_M-t} q_\lambda(x)dxdt. \end{aligned} \tag{12}$$

The observation process  $\tilde{Y}(t)$  then becomes

$$d\tilde{Y}(t) = -\lambda(t)H(t)q_\lambda dt + \frac{1}{2}F(t)dt + H(t)dw(t, \cdot), \tag{13}$$

where

$$\begin{aligned} F &= \left[ \int_{T_m-t}^{T_M-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2)dx_1dx_2 + 2 \int_0^{T_m-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2)dx_1dx_2 \right] \\ H(t)[\cdot] &= \left[ \int_{T_m-t}^{T_M-t} (\cdot)dx \right]. \end{aligned}$$

We now construct the augmented observation process  $\vec{Y}(t) = [Y(t), \tilde{Y}(t)]'$  and this satisfies

$$d\vec{Y}(t) = \vec{H}_\delta f(t, \cdot)dt - \lambda(t)\vec{H}(t)q_\lambda dt + \frac{1}{2}\vec{F}(t)dt + \vec{H}(t)dw(t, \cdot), \tag{14}$$

where

$$\vec{H}_\delta[\cdot] = \begin{bmatrix} H_\delta[\cdot] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[ \frac{1}{\tau_i} \int_G (\delta(x - \tau_i) - \delta(x))(\cdot)dx \right]_{m \times 1} \\ 0 \end{bmatrix}_{(m+1) \times 1},$$

and

$$\vec{F} = \begin{bmatrix} \left[ \frac{1}{\tau_i} \tilde{q}(\tau_i) \right]_{m \times 1} \\ \int_{T_m-t}^{T_M-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2) dx_1 dx_2 + 2 \int_0^{T_m-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2) dx_1 dx_2 \end{bmatrix}_{(m+1) \times 1}$$

$$\vec{H}(t)q_\lambda = \begin{bmatrix} \left[ \frac{1}{\tau_i} \int_0^{\tau_i} q_\lambda dx \right]_{m \times 1} \\ \int_{T_m-t}^{T_M-t} q_\lambda dx \end{bmatrix}_{(m+1) \times 1}.$$

Note that the observation noise covariance becomes

$$\vec{\Phi}(t) = \begin{bmatrix} \frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} q(x_1, x_2) dx_1 dx_2 & \frac{1}{\tau_i} \int_0^{\tau_i} \int_{T_m-t}^{T_M-t} q(x_1, x_2) dx_1 dx_2 \\ \frac{1}{\tau_j} \int_{T_m-t}^{T_M-t} \int_0^{\tau_j} q(x_1, x_2) dx_1 dx_2 & \int_{T_m-t}^{T_M-t} \int_{T_m-t}^{T_M-t} q(x_1, x_2) dx_1 dx_2 \end{bmatrix}_{(m+1) \times (m+1)}.$$

We can indeed show that  $\vec{\Phi}(t)$  is invertible [3]. Hence, without adding the artificial observation noise, we can derive the Kalman filter equation for the augmented observation  $\vec{Y}$  (see [3] for detailed derivations)

$$d \begin{pmatrix} \hat{f}(t, x) \\ \hat{\lambda}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{f}(t, x)}{\partial x} - q_\lambda(x) \hat{\lambda}(t) \\ a \hat{\lambda}(t) \end{pmatrix} dt + \begin{pmatrix} \frac{1}{2} \frac{d\tilde{q}(x)}{dx} \\ b \end{pmatrix} dt$$

$$+ \begin{pmatrix} \vec{P}(t) \begin{pmatrix} \vec{H}_\delta^* \\ -\vec{H}^*(q_\lambda) \end{pmatrix} + \begin{pmatrix} \vec{H}^*(q) \\ (\sigma_\lambda, \vec{H}^*(q)) \end{pmatrix} \end{pmatrix} \vec{\Phi}^{-1} d\vec{\ell}(t), \quad (15)$$

where the innovation process  $\vec{\ell}(t) = [\ell(t) \ \tilde{\ell}(t)]^*$  is defined by

$$\begin{bmatrix} \ell(t) \\ \tilde{\ell}(t) \end{bmatrix} = \begin{bmatrix} Y(t) - Y(0) - \int_0^t \left( H_\delta \hat{f} - \hat{\lambda}(s) H q_\lambda + \frac{1}{2} F \right) ds \\ \tilde{Y}(t) - \tilde{Y}(0) - \int_0^t \left\{ \frac{1}{2} \bar{q}(s) + \bar{q}_2(s) - \hat{\lambda}(s) \bar{q}_\lambda(s) \right\} ds \end{bmatrix}, \quad (16)$$

$$\bar{q}(s) = \int_{T_m-s}^{T_M-s} \int_{T_m-s}^{T_M-s} q(x_1, x_2) dx_1 dx_2, \quad (17)$$

$$\bar{q}_2(s) = \int_0^{T_m-s} \int_{T_m-s}^{T_M-s} q(x_1, x_2) dx_1 dx_2, \quad (18)$$

$$\bar{q}_\lambda(s) = \int_{T_m-s}^{T_M-s} q_\lambda(x) dx, \quad (19)$$

and

$$\vec{P}(t) = \begin{pmatrix} \int_G p_{ff}(t, x, y) (\cdot) dy & p_{f\lambda}(t, x) \\ p_{f\lambda}(t, x) & p_{\lambda\lambda}(t) \end{pmatrix},$$

and where

$$\frac{\partial p_{ff}(t, x, y)}{\partial t} = \frac{\partial p_{ff}(t, x, y)}{\partial x} + \frac{\partial p_{ff}(t, x, y)}{\partial y} - q_\lambda(x) p_{\lambda f}(t, y) - p_{f\lambda}(t, x) q_\lambda(y)$$

$$- \left[ \frac{p_{ff}(t, x, \tau_i) - p_{ff}(t, x, 0)}{\tau_i} - p_{f\lambda}(t, x) \frac{1}{\tau_i} \int_0^{\tau_i} q_\lambda(z) dz + \frac{1}{\tau_i} \int_0^{\tau_i} q(x, z) dz \right]_{1 \times m}$$

$$\times \Phi^{-1} \left[ \frac{p_{ff}(t, \tau_j, y) - p_{ff}(t, 0, y)}{\tau_j} - p_{f\lambda}(t, y) \frac{1}{\tau_j} \int_0^{\tau_j} q_\lambda(z) dz + \frac{1}{\tau_j} \int_0^{\tau_j} q(x, z) dz \right]_{m \times 1}$$

$$+ q(x, y).$$

$$\begin{aligned} \frac{\partial p_{f\lambda}(t, x)}{\partial t} &= \frac{\partial p_{f\lambda}(t, x)}{\partial x} - q_\lambda(x)p_{\lambda\lambda}(t) + ap_{f\lambda}(t, x) \\ &\quad - \left[ \frac{p_{ff}(t, x, \tau_i) - p_{ff}(t, x, 0)}{\tau_i} - p_{f\lambda}(t, x) \frac{1}{\tau_i} \int_0^{\tau_i} q_\lambda(z)dz + \frac{1}{\tau_i} \int_0^{\tau_i} q(x, z)dz \right]_{1 \times m} \Phi^{-1} \\ &\quad \times \left[ \frac{p_{f\lambda}(t, \tau_j) - p_{f\lambda}(t, 0)}{\tau_j} - p_{\lambda\lambda}(t) \frac{1}{\tau_j} \int_0^{\tau_j} q_\lambda(z)dz + \frac{1}{\tau_j} \int_0^{\tau_j} \int_G \sigma_\lambda(z)q(z, y)dzdy \right]_{m \times 1} \\ &\quad + \int_G \sigma_\lambda(y)q(y, x)dy. \\ \frac{dp_{\lambda\lambda}(t)}{dt} &= 2ap_{\lambda\lambda}(t) \\ &\quad - \left[ \frac{p_{f\lambda}(t, \tau_i) - p_{f\lambda}(t, 0)}{\tau_i} - p_{\lambda\lambda}(t) \frac{1}{\tau_i} \int_0^{\tau_j} q_\lambda(z)dz + \frac{1}{\tau_i} \int_0^{\tau_i} \int_G \sigma_\lambda(z)q(z, y)dzdy \right]_{1 \times m} \Phi^{-1} \\ &\quad \times \left[ \frac{p_{f\lambda}(t, \tau_j) - p_{f\lambda}(t, 0)}{\tau_j} - p_{\lambda\lambda}(t) \frac{1}{\tau_j} \int_0^{\tau_j} q_\lambda(z)dz + \frac{1}{\tau_j} \int_0^{\tau_j} \int_G \sigma_\lambda(z)q(z, y)dzdy \right]_{m \times 1} \\ &\quad + \int_G \int_G \sigma_\lambda(x)q(x, y)\sigma_\lambda(y)dxdy. \end{aligned}$$

**5. Adaptive Filtering.** For applying the filtering algorithm established here to the hedging problem, we need to identify the systems parameters in (4). In this paper, we propose the recursive algorithm for estimating the risk premium and the associated parameters by using the parallel filtering algorithm in [8, 10]. For simplicity, we restrict ourselves as

$$q_\lambda(x) = \frac{1}{2} \frac{d\tilde{q}(x)}{dx}, \text{ and } (\sigma_\lambda, \cdot) = \sigma_\ell(1, \cdot).$$

Now assuming that  $\tilde{q}(x)$  is identified from the method stated in Section 3, we estimate the system parameters  $a, b$  and a vector  $\sigma_\ell$  in (4) as  $\theta$  in some bounded set  $D \subset R^3$ . To apply the parallel filtering algorithm, we assume that  $D$  is a large finite set. To obtain these values, we apply the generating procedure as used in the particle filter algorithm [2, 6]. We generate  $\theta_i, i = 1, 2, \dots, N$  from the uniform random distribution with some upper and lower bounds. We define

$$\hat{\lambda}^i(t) = E\{\lambda(t) | \mathcal{Y}_t^{\vec{\theta}}, \theta = \theta_i\} \tag{20}$$

$$\hat{f}^i(t, x) = E\{f(t, x) | \mathcal{Y}_t^{\vec{\theta}}, \theta = \theta_i\} \tag{21}$$

where  $\hat{\lambda}^i(t)$  and  $\hat{f}^i(t, x)$  can be computed on-line from the conditional filter covariance equations in Section 4 and the filter Equation (15) for tuned to  $\theta_i$ , respectively.

The application of Bayes' rule yields

$$P(\theta_i | \mathcal{Y}_t^{\vec{\theta}}) = \frac{p(\vec{Y}_t | \theta_i)}{\sum_{i=1}^N p(\vec{Y}_t | \theta_i)}, \tag{22}$$

where  $p(\vec{Y}_t | \theta_i)$  is a likelihood function given by

$$\begin{aligned} p(\vec{Y}_t | \theta_i) &= \frac{1}{\left( (2\pi)^{m+1} \det(\vec{\Phi}) \right)^{\frac{1}{2}}} \exp \left\{ \frac{1}{2} \int_0^t \left( \vec{H}_\delta \hat{f}^i(s, \cdot) - \hat{\lambda}^i(s) \vec{H}(s) + \frac{1}{2} \vec{F}(s) \right)^* \vec{\Phi}^{-1} d\vec{Y}(s) \right. \\ &\quad \left. - \int_0^t \left| \vec{\Phi}^{-1/2} \left( \vec{H}_\delta \hat{f}^i(s, \cdot) - \hat{\lambda}^i(s) \vec{H}(s) \right) + \frac{1}{2} \vec{F}(s) \right|^2 ds \right\} \end{aligned} \tag{23}$$

and we used that the initial distribution of  $\theta$  is uniform. Hence, we get

$$\hat{\theta}(t) = \sum_{i=1}^N \theta_i P(\theta_i | \mathcal{Y}_t) \tag{24}$$

$$\hat{\lambda}(t) = \sum_{i=1}^N \hat{\lambda}^i(t) P(\theta_i | \mathcal{Y}_t) \tag{25}$$

$$\hat{f}(t, x) = \sum_{i=1}^N \hat{f}^i(t, x) P(\theta_i | \mathcal{Y}_t). \tag{26}$$

Theoretically the parallel algorithm generates the optimal estimates on-line. However, in practice there are many cases that the estimates of unknown parameter  $\theta$  are not sensitive to the innovation process. In this paper we suggest using the resampling method to avoid this insensitivity property as was often used in particle filters [2, 6]. Now we list up the whole scheme of our parallel filtering with the forced resampling method:

**Parallel Filtering Algorithm**

- Generate N particles for  $\theta$ .
- Solve the Kalman filter (15) for each  $\theta = \theta_i$ .
- Get  $P(\theta_i | \mathcal{Y}_t)$  from (22) and its cumulative probability.
- At the time  $t = m\Delta t$  for some  $m$  and  $\Delta t$ , we generate N uniformly distributed numbers in  $[0, 1]$ . From these random numbers and the cumulative probability, we find the important particles as illustrated in Figure 1. (Resampling)

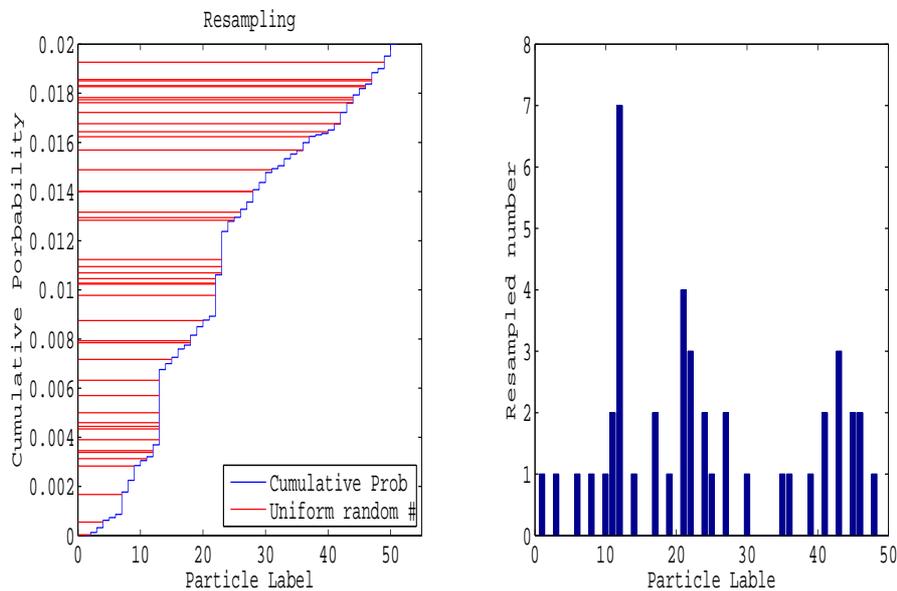


FIGURE 1. Schematic procedure for resampling

- The optimal estimates  $\hat{\lambda}$ ,  $\hat{f}$  and  $\hat{\theta}$  can be obtained form (24), (25) and (26).

6. **Simulation Studies.** In this digital simulation study, from [3] we set

$$q(x_1, x_2) = \sigma^2 \sum_{i=1}^{20} \frac{1}{i^2} \exp(-cx_1) \sin\left(\frac{\pi i x_1}{30}\right) \exp(-cx_2) \sin\left(\frac{\pi i x_2}{30}\right) + \sigma_r^2 \exp(-a_r(x_1 + x_2)).$$

The system parameters are given in Table 1.

TABLE 1. System parameters

$\sigma$	$c$	$a_r$	$\sigma_r$	$\sigma_\ell$	$a$	$b$
0.6269	0.1627	3.3114	0.2949	0.15	-2	1

To simulate the yield curve and bond data, we used the parameters for the yield and bond data as shown in Table 2.

TABLE 2. Yield and bond parameters

$\tau_1$	$\tau_2$	$\tau_3$	$\tau_4$	$\tau_5$	$\tau_6$	$\tau_7$	$T_m$	$T_M$
1	2	3	5	7	10	20	0.5	0.75

Now we generated the yield and bond data. The yield curve  $[y_1, \dots, y_7]$  and  $\log P(:, T_M)/P(:, T_m)$  are shown in Figures 2 and 3, respectively.

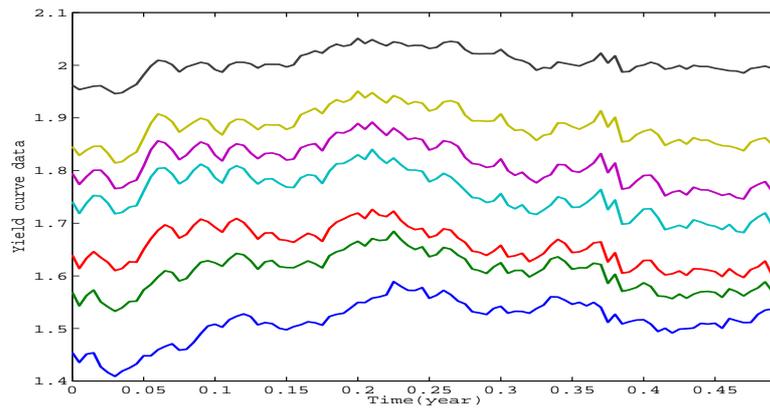


FIGURE 2. Yield curve data

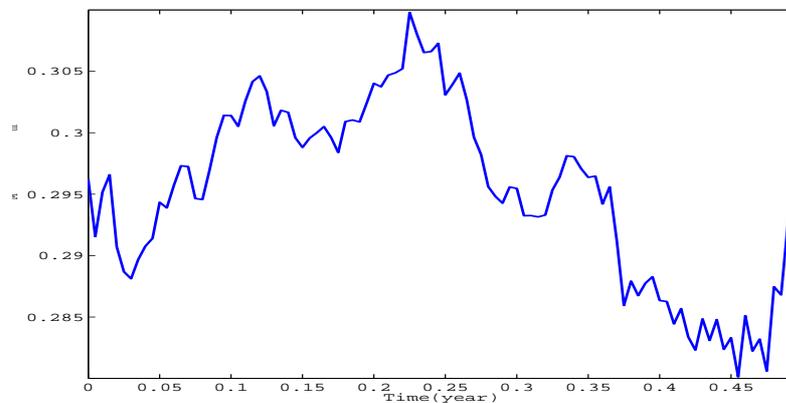


FIGURE 3.  $\log \frac{P(:, T_M)}{P(:, T_m)}$  data

Assuming  $\sigma$ ,  $c$ ,  $a_r$  and  $\sigma_r$  are known, we set  $\sigma_\ell$ ,  $a$  and  $b$  as unknown parameters. These upper and lower bounds are chosen as

$$\left. \begin{aligned} 0.05 &\leq \sigma_\ell \leq 0.2 \\ -3 &\leq a \leq -1 \\ 0 &\leq b \leq 2 \end{aligned} \right\} \tag{27}$$

We generate 50 candidates  $\theta_i, i = 1, 2, \dots, 80$  for  $\theta = [a \ b \ \sigma_\ell]$  from the uniform distribution with the bounds given by (27). For performing the on-line algorithm established here, we use the forced resampling method. In this simulation, we made resampling for every  $5\Delta t$  period for  $\Delta t = 0.001$ .

We show the results for estimating the stochastically-varying risk-premium in Figure 4.

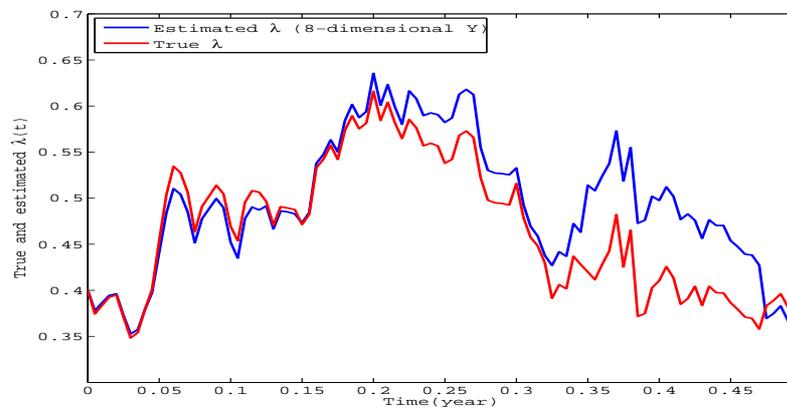


FIGURE 4. True and estimated  $\lambda(t)$

We also present the estimate of  $f(t, x)$  in Figure 5.

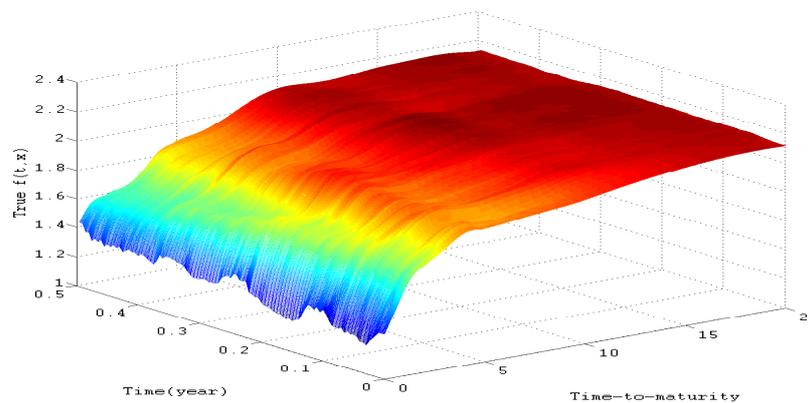
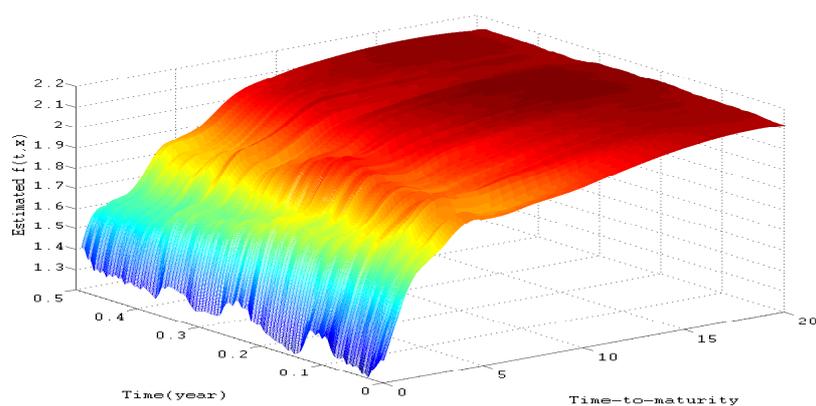
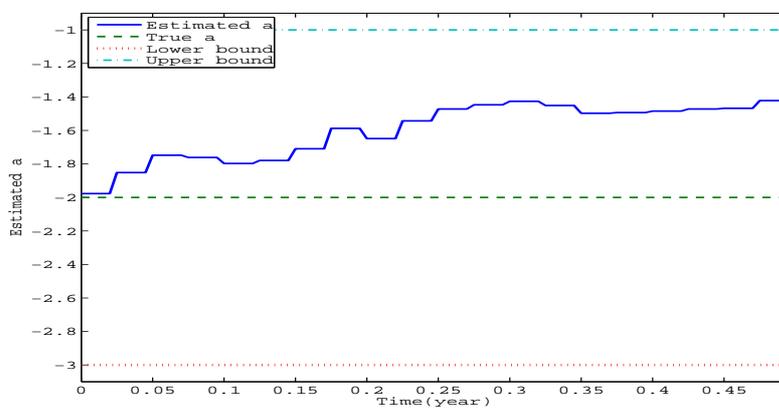
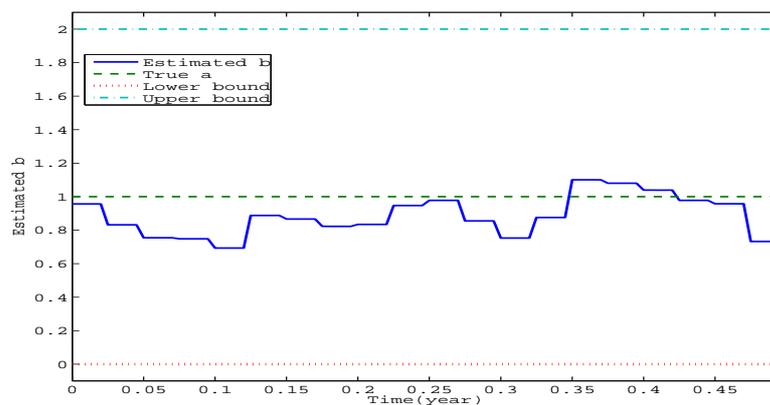


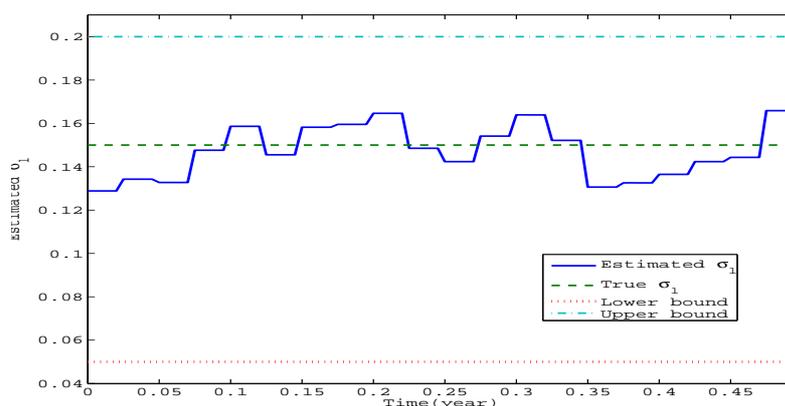
FIGURE 5. Estimated  $f(t, x)$

We also present the true value of  $f(t, x)$  in Figure 6.

Now we shall present on-line parameter estimates for  $a$ ,  $b$  and  $\sigma_\ell$  in Figures 7-9, respectively.

FIGURE 6. True  $f(t, x)$ FIGURE 7. Estimated  $a$ FIGURE 8. Estimated  $b$ 

At the resampled timing, the estimates for unknown parameters have jumps and these jumps improve the estimate  $\hat{\lambda}(t)$ . If we do not use the forced resampling, the estimation results become worse.

FIGURE 9. Estimated  $\sigma_\ell$ 

**7. Conclusions.** We proposed the on-line estimation procedure for the stochastically moving risk-premium and the systems parameters by using the yield and bond data which are used for hedging some option claims. Hence, the estimation method developed here can be directly applied to the mean-variance hedging problem in incomplete markets.

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